Signals and Systems:
Theory and Applications

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Exercise 1-1 If signal $y(t)$ is obtained from $x(t)$ by applying the transformation $y(t) = x(-4t-8)$, determine the values of the transformation parameters a and T.

Solution: From Eq. (1.5), $y(t) = x(at-b) = x(a(t-\frac{b}{a})) = x(a(t-T))$ where $T = \frac{b}{a}$.

Here, $y(t) = x(-4t-8) = x(-4(t+2))$, so $\boxed{a = -4 \text{ and } T = -2.}$

Exercise 1-2 If $x(t) = t^3$ and $y(t) = 8t^3$, are $x(t)$ and $y(t)$ related by a transformation?

Solution: $\boxed{\text{Yes,}}$ since $x(2t) = (2t)^3 = 8t^3 = y(t)$.

Even though $y(t) = 8x(t)$, this is not a transformation as defined in Section 1-2.

Exercise 1-3 What types of transformations connect $x(t) = 4t$ to $y(t) = 2(t+4)$?

Solution: Let $y(t) = x(a(t-T))$ for a time-scaling transformation with factor a and a time-shift transformation with a time delay of T.

Since $x(t) = 4t$, we have $y(t) = 4[a(t-T)] = 4at - 4aT$. We want $y(t) = 2(t+4) = 2t+8$.

So $4a = 2$ and $-4aT = 8$. This yields $\boxed{a = \frac{1}{2} \text{ and } T = -4.}$

Then

$$y(t) = x(a(t-T)) = x\left(\frac{1}{2}(t+4)\right) = x\left(\frac{t}{2}+2\right).$$

Even though $y(t) = 2x(t) + 8$, this is not a transformation as defined in Section 1-2.

Exercise 1-4 Which of the following functions have even-symmetrical waveforms, odd-symmetrical waveforms, or neither? (a) $x_1(t) = 3t^2$, (b) $x_2(t) = \sin(2t)$, (c) $x_3(t) = \sin^2(2t)$, (d) $x_4(t) = 4e^{-t}$, (e) $x_5(t) = |\cos 2t|$.

Solution: A function $x(t)$ has an even-symmetrical waveform if $x(-t) = x(t)$. It is symmetric about the vertical axis.

A function $x(t)$ has an odd-symmetrical waveform if $x(-t) = -x(t)$. Reflecting an odd-symmetric function about the vertical axis, then about the horizontal axis (or vice-versa), leaves it unaltered.

(a) $x_1(-t) = 3(-t)^2 = 3t^2 = x_1(t)$, so $x_1(t)$ has an $\boxed{\text{even-symmetrical waveform.}}$

(b) $x_2(-t) = \sin(-2t) = -\sin(2t) = -x_2(t)$,

so $x_2(t)$ has an $\boxed{\text{odd-symmetrical waveform.}}$

(c) $x_3(-t) = \sin^2(-2t) = (-\sin(2t))^2 = \sin^2(2t) = x_3(t)$,

so $x_3(t)$ has an $\boxed{\text{even-symmetrical waveform.}}$

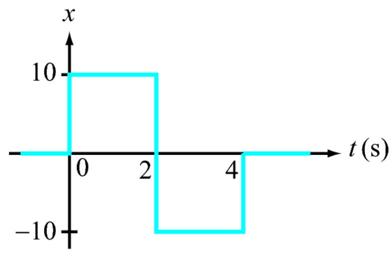
(d) $x_4(-t) = 4e^t \neq \pm x_4(t)$, so the waveform of $x_4(t)$ has $\boxed{\text{no symmetry.}}$

(e) $x_5(-t) = |\cos(-2t)| = |\cos(2t)| = x_5(t)$,

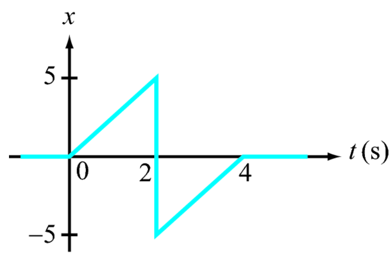
so $x_5(t)$ has an $\boxed{\text{even-symmetrical waveform.}}$

Note that $\cos(2t)$ also has an even-symmetrical waveform.

Exercise 1-5 Express the waveforms shown in Fig. E1-5 in terms of unit step functions.



(a)



(b)

Figure E1-5

Solution: See Fig. E1-5(a); (a) is shown in the left column; (b) is shown in the right column.

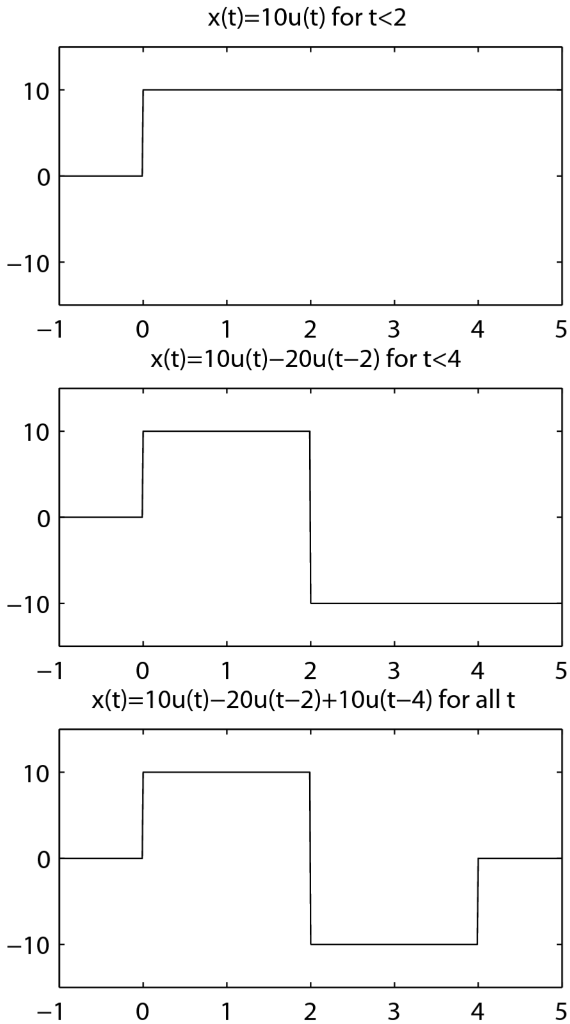


Figure E1-5(a)

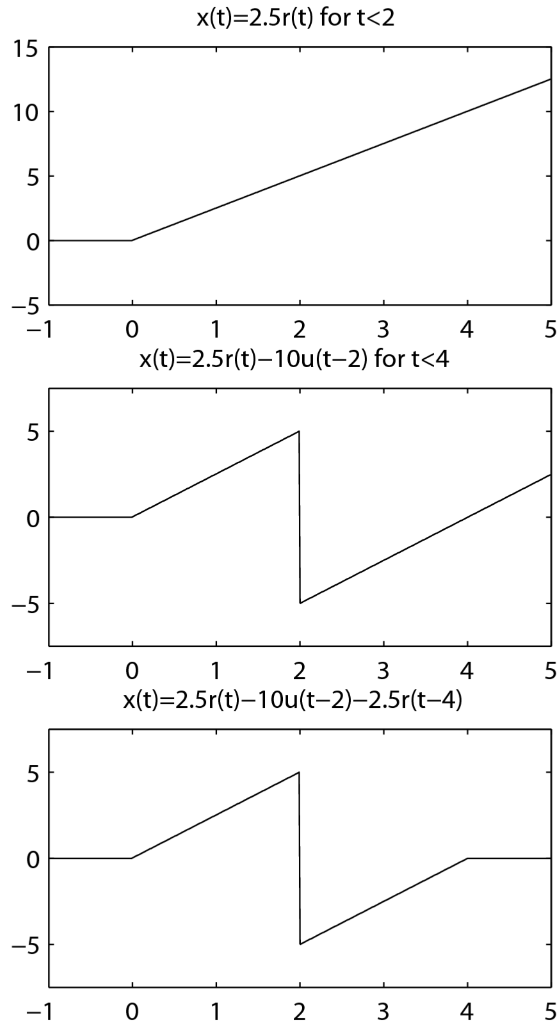


Figure E1-5(b)

(a) $x(t)$ starts at 0, jumps to 10 at $t = 0$, stays there until $t = 2$. Hence, $x(t) = 10u(t)$ for $t < 2$. $x(t)$ drops from 10 to –10 at $t = 2$, stays until $t = 4$. Hence, $x(t) = 10u(t) - 20u(t-2)$ for $t < 4$. $x(t)$ jumps from –10 to 0 at $t = 4$ and stays. FINAL:

$$\boxed{x(t) = 10u(t) - 20u(t-2) + 10u(t-4).}$$

(b) $x(t)$ starts at 0, increases with slope $\frac{5}{2} = 2.5$ until $t = 2$. Hence, $x(t) = 2.5r(t)$ for $t < 2$. $x(t)$ drops 5 to -5 at $t = 2$, then increases with slope 2.5. $x(t) = 2.5r(t) - 10u(t-2)$ for $t < 4$. $x(t)$ levels off at 0 at $t = 4$. FINAL:

$$\boxed{x(t) = 2.5r(t) - 10u(t-2) - 2.5r(t-4).}$$

Exercise 1-6 How is $u(t)$ related to $u(-t)$?

Solution: $u(-t)$ is simply $u(t)$ reflected about the vertical axis. So

$$\boxed{u(t) \text{ and } u(-t) \text{ are mirror images of one another.}}$$

This is true for *any* function $x(t)$, not just $u(t)$.

Exercise 1-7 If $x(t)$ is the rectangular pulse shown in Fig. E1-7(a), determine its time derivative $x'(t)$ and plot it.

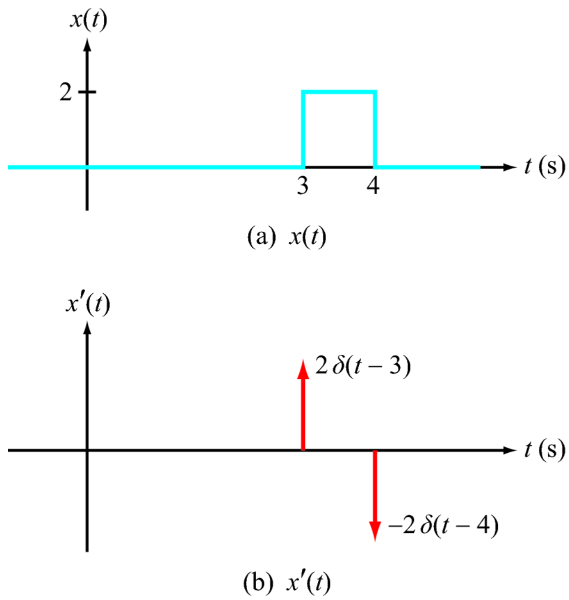


Figure E1-7

Solution: In Fig. E1-7(a), $x(t) = 2u(t-3) - 2u(t-4)$. From Eq. (1.25a),

$$\frac{d}{dt}\,[u(t-T)] = \delta(t-T).$$

So

$$x'(t) = \frac{dx}{dt} = \frac{d}{dt}\,[2u(t-3) - 2u(t-4)] = \boxed{2\delta(t-3) - 2\delta(t-4).}$$

Exercise 1-8 The radioactive decay equation for a certain material is given by $n(t) = n_0 e^{-t/\tau}$, where n_0 is the initial count at $t = 0$. If $\tau = 2 \times 10^8$ s, how long is its half-life?

Solution: The half-life is the time $t_{1/2}$ at which

$$\frac{n(t_{1/2})}{n(0)} = \frac{1}{2}.$$

So $t_{1/2}$ solves

$$\frac{1}{2} = \frac{n(t_{1/2})}{n(0)} = e^{-t_{1/2}/2\times 10^8}.$$

Solving this equation gives $t_{1/2} = -(2 \times 10^8) \log\left(\frac{1}{2}\right) = 1.386 \times 10^8 \text{ s} \approx \boxed{\text{4 years.}}$

Exercise 1-9 If the current $i(t)$ through a resistor R decays exponentially with a time constant τ, what is the ratio of the power dissipated in the resistor at time $t = \tau$ to its value at $t = 0$?

Solution: The current is $i(t) = i(0)\, e^{-t/\tau}$. The power is $p(t) = i^2(t)\, R = i^2(0)\, Re^{-2t/\tau}$. So $p(0) = i^2(0)\, R$.

The ratio of powers is

$$\frac{p(\tau)}{p(0)} = \frac{i^2(0)\, Re^{-2\tau/\tau}}{i^2(0)\, R} = e^{-2} = \boxed{0.135.}$$

Exercise 1-10 Determine the values of P_{av} and E for a pulse signal given by

$$x(t) = 5 \text{ rect}\left(\frac{t-3}{4}\right).$$

Solution: $x(t) = \begin{cases} 5 & \text{for } 1 < t < 5, \\ 0 & \text{otherwise.} \end{cases}$

So

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{1}^{5} |5|^2 \, dt = 100.$$

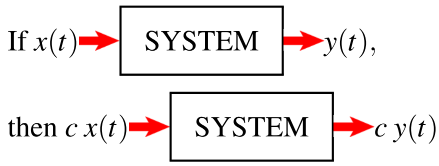
Since E is finite, $\boxed{P_{\text{av}} = 0.}$

Note that E is invariant to time shifts, so we could have used

$$\boxed{E = \int_{-2}^{2} |5|^2 \, dt = 100.}$$

Exercise 2-1 Does the system $y(t) = x^2(t)$ have the scaling property?

Solution: The scaling property of a system is:



for any constant c.

The response to $[c\,x(t)]$ is the output $[c\,x(t)]^2 = c^2\,x^2(t) = c^2\,y(t) \neq c\,y(t)$.

So the system **does not have the scaling property.**

Exercise 2-2 Which of the following systems is linear?

(a) $y_1(t) = |\sin(3t)|\, x(t)$.

(b) $y_2(t) = a\frac{dx}{dt}$.

(c) $y_3(t) = |x(t)|$.

(d) $y_4(t) = \sin[x(t)]$.

Solution: A system is linear if it has both the scaling property and the additivity property. The scaling property is:

If $x(t) \rightarrow$ SYSTEM $\rightarrow y(t)$,

then $c\,x(t) \rightarrow$ SYSTEM $\rightarrow c\,y(t)$

for any constant c. The additivity property is:

If $x_1(t) \rightarrow$ SYSTEM $\rightarrow y_1(t)$

and $x_2(t) \rightarrow$ SYSTEM $\rightarrow y_2(t)$,

then $[x_1(t) + x_2(t)] \rightarrow$ SYSTEM $\rightarrow [y_1(t) + y_2(t)]$.

(a) The system is $y(t) = |\sin(3t)|\, x(t)$.
The response to $[cx(t)]$ is the output $|\sin(3t)|[c\,x(t)] = c|\sin(3t)|\, x(t) = c\,y(t)$.

So the system has the scaling property.

- $x_1(t) \rightarrow$ SYSTEM $\rightarrow y_1(t) = |\sin(3t)|\, x_1(t)$.
- $x_2(t) \rightarrow$ SYSTEM $\rightarrow y_2(t) = |\sin(3t)|\, x_2(t)$.
- $[x_1(t) + x_2(t)] \rightarrow$ SYSTEM $\rightarrow |\sin(3t)|[x_1(t) + x_2(t)]$.

The response to $[x_1(t) + x_2(t)]$ is the output

$$|\sin(3t)|[x_1(t) + x_2(t)] = |\sin(3t)| x_1(t) + |\sin(3t)|\, x_2(t) = y_1(t) + y_2(t).$$

So the system has the additivity property.

Since the system has both the scaling and additivity properties, it is linear.

(b) The system is $y(t) = a\,\frac{dx}{dt}$.
The response to $[c\,x(t)]$ is the output

$$a\frac{d(cx)}{dt} = ac\frac{dx}{dt} = c\,y(t).$$

So the system has the scaling property.

- $x_1(t)$ → SYSTEM → $y_1(t) = a\dfrac{dx_1}{dt}$.
- $x_2(t)$ → SYSTEM → $y_2(t) = a\dfrac{dx_2}{dt}$.
- $[x_1(t) + x_2(t)]$ → SYSTEM → $a\dfrac{d}{dt}[x_1(t) + x_2(t)]$.

The response to $[x_1(t) + x_2(t)]$ is the output

$$a\frac{d}{dt}[x_1(t) + x_2(t)] = a\frac{dx_1}{dt} + a\frac{dx_2}{dt} = y_1(t) + y_2(t).$$

So the system has the additivity property.

Since the system has both the scaling and additivity properties, **it is linear.**

(c) The system is $y(t) = |x(t)|$.
We show that the system does not have the scaling property.
Trying $c = -1$ shows that the response to $-x(t)$ is the output

$$|-x(t)| = |x(t)| = y(t) \neq -y(t).$$

The system does not have the scaling property. So the system **is not linear.**

(d) The system is $y(t) = \sin[x(t)]$.
We show that the system does not have the scaling property.
Trying $c = 2$ shows that the response to $2x(t)$ is the output

$$\sin[2x(t)] \neq 2\sin[x(t)] = 2y(t).$$

The system does not have the scaling property. So the system **is not linear.**

Exercise 2-3 Which systems are time-invariant?

(a) $y(t) = \frac{dx}{dt} + \sin[x(t-1)]$.

(b) $\frac{dy}{dt} = 2\sin[x(t-1)] + 3\cos[x(t-1)]$?

Solution: A system is time-invariant if it has the property that:

If $x(t) \rightarrow$ SYSTEM $\rightarrow y(t)$,

then $x(t-T) \rightarrow$ SYSTEM $\rightarrow y(t-T)$

for any constant T.

(a) The system is

$$y(t) = \frac{dx}{dt} + \sin[x(t-1)].$$

The response to $x(t-T)$ is the output

$$\frac{d(x(t-T))}{dt} + \sin[x(t-T-1)] = y(t-T).$$

So the system **is time-invariant.**

(b) The system is

$$\frac{dy}{dt} = 2\sin[x(t-1)] + 3\cos[x(t-1)].$$

Substituting $x(t-T)$ for $x(t)$ and $y(t-T)$ for $y(t)$ gives

$$\frac{dy(t-T)}{dt} = 2\sin[x(t-T-1)] + 3\cos[x(t-T-1)],$$

which is the system with t replaced with $t-T$.

So the system **is time-invariant.**

Exercise 2-4 Determine the impulse response of a system whose step response is

$$y_{\text{step}}(t) = \begin{cases} 0, & t \le 0 \\ t, & 0 \le t \le 1 \\ 1, & t \ge 1. \end{cases}$$

Solution: The impulse response is the derivative of the step response:

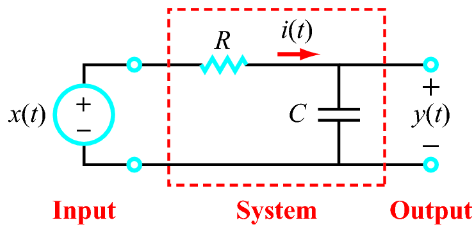
$$h(t) = \frac{dy_{\text{step}}}{dt}.$$

For the given $y_{\text{step}}(t)$, we have:

$$h(t) = \frac{dy_{\text{step}}}{dt} = \begin{cases} \dfrac{d0}{dt}, & t \le 0 \\ \dfrac{dt}{dt}, & 0 \le t \le 1 \\ \dfrac{d1}{dt}, & t \ge 1 \end{cases} = \begin{cases} 0, & t \le 0 \\ 1, & 0 \le t \le 1 \\ 0, & t \ge 1 \end{cases}$$

This rectangular pulse can be written succinctly as $\boxed{h(t) = u(t) - u(t-1).}$

Exercise 2-5 The RC circuit of Fig. 2-5(a) is excited by $x(t) = (1 - 1000t)[u(t) - u(t - 0.001)]$.



(a) RC circuit

Figure 2-5(a)

Compute the capacitor voltage $y(t)$ for $t > 0.001$ s, given that $\tau_c = 1$ s.

Solution: $x(t)$ is a very short wedge-shaped pulse (Fig. E2-5). Its duration of 0.001 s is much less than $\tau_c = 1$ s.

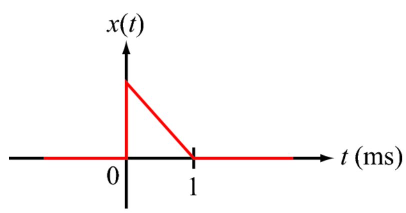


Figure E2-5

So the response $y(t)$ of the RC circuit to $x(t)$ will be, for $t > 0.001$ s, the same as its response to an impulse having the same area as the pulse. The area under the wedge-shaped pulse is $\frac{1}{2}(1)(0.001) = 0.0005$. The impulse response of the RC circuit is

$$h(t) = \frac{1}{\tau_c}\, e^{-t/\tau_c}\, u(t) = e^{-t}\, u(t).$$

So the response $y(t)$ is, for $t > 0.001$, $y(t) = 0.0005e^{-t}\, u(t)$.

Exercise 2-6 Apply graphical convolution to the waveforms of $x(t)$ and $h(t)$ shown in Fig. E2-6 to determine $y(t) = h(t) * x(t)$.

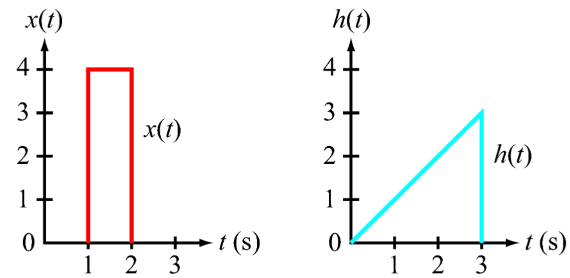


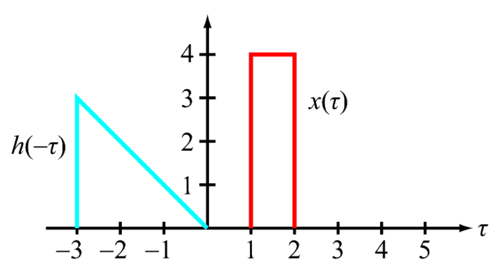
Figure E2-6

Solution:

$$x(\tau) = \begin{cases} 4 & \text{for } 1 < \tau < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$h(\tau) = \begin{cases} \tau & \text{for } 0 < \tau < 3 \\ 0 & \text{otherwise} \end{cases}$$

Note that $0 < (t - \tau) \Longrightarrow \tau < t$ and $(t - \tau) < 3 \Longrightarrow (t - 3) < \tau$.



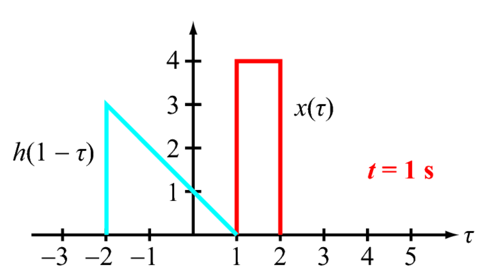
(a) At $t = 0$, overlap = 0

Figure E2-6(a)

For $t \leq 1$:

$$\int_{-\infty}^{\infty} x(\tau)\, h(t - \tau)\, d\tau = 0$$

(there is no overlap).



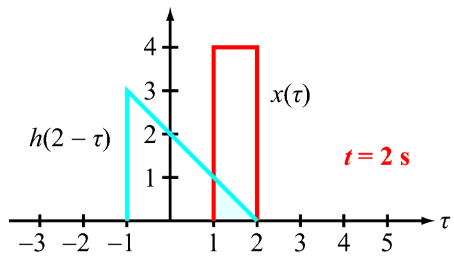
(b) At $t = 1$ s, overlap = 0

Figure E2-6(b)

For $1 \le t \le 2$:

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{1}^{t} 4(t-\tau)\, d\tau = 2t^2 - 4t + 2.$$

This is the area of the blue triangle in Fig. E2-6(c).



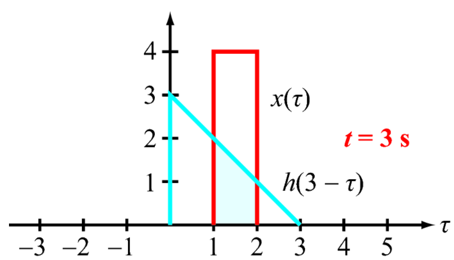
(c) At t = 2 s, overlap = 1/2 × 4 = 2

Figure E2-6(c)

For $2 \le t \le 4$:

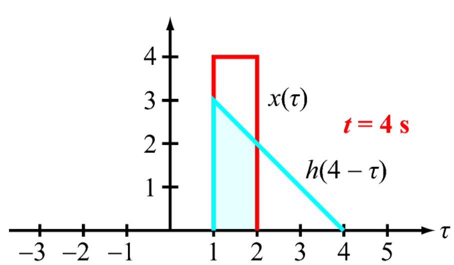
$$\int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = \int_{1}^{2} 4(t-\tau)\, d\tau = 4t - 6.$$

This is the area of the blue triangle in Fig. E2-6(d) and Fig. E2-6(e).



(d) At t = 3 s, overlap = 1.5 × 4 = 6

Figure E2-6(d)



(e) At t = 4 s, overlap = 2.5 × 4 = 10

Figure E2-6(e)

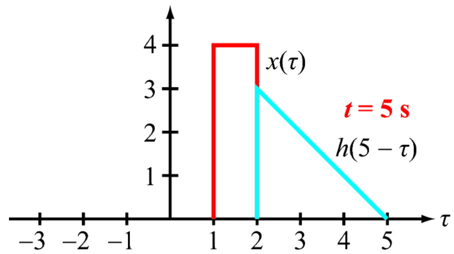
For $4 \le t \le 5$:

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{t-3}^{2} 4(t-\tau)\, d\tau = -2t^2 + 8t + 10.$$

For $t \ge 5$:

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = 0$$

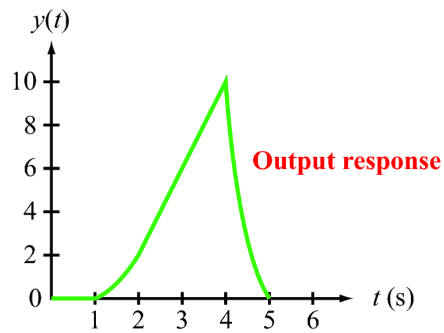
(there is no overlap).



(f) At $t = 5$ s, overlap = 0

Figure E2-6(f)

Check: If there are no impulses, the result of a convolution should be a continuous waveform. The above expressions agree at the endpoints $t = 1, 2, 4, 5$ of each of the above intervals. The result is plotted in Fig. E2-6(g). The maximum is $y(4) = 10$.



(g) $y(t)$

Figure E2-6(g)

Exercise 2-7 Evaluate $u(t) * \delta(t-3) - u(t-4) * \delta(t+1)$.

Solution: Using convolution property #6 in Table 2-1, $u(t) * \delta(t-3) = u(t-3)$ and $u(t) * \delta(t) = u(t)$. Then using convolution property #5 with $T_1 = 4$ and $T_2 = -1$, we obtain

$$u(t-4) * \delta(t+1) = u(t-4+1) = u(t-3).$$

So

$$u(t) * \delta(t-3) - u(t-4) * \delta(t+1) = u(t-3) - u(t-3) = \boxed{0.}$$

Exercise 2-8 Evaluate $\lim_{t\to\infty}[e^{-3t}\,u(t)*u(t)]$.

Solution: Let $y(t) = e^{-3t}\,u(t)*u(t)$.

Using convolution property #9 in Table 2-1,

$$y(t) = \int_{-\infty}^{t} e^{-3\tau}\,u(\tau)\,d\tau = \int_{0}^{t} e^{-3\tau}\,d\tau = \frac{1}{3}(1-e^{-3t})\,u(t).$$

Then

$$\lim_{t\to\infty}(e^{-3t}\,u(t)*u(t)) = \lim_{t\to\infty} y(t) = \boxed{\frac{1}{3}\,.}$$

Exercise 2-9 A system's impulse response is $h(t) = u(t-1)/t^2$. Is the system BIBO stable?

Solution: The system is BIBO stable if and only if it is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

is finite. Here,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| \frac{u(t-1)}{t^2} \right| dt = \int_{1}^{\infty} \frac{dt}{t^2} = \left. \frac{-1}{t} \right|_{1}^{\infty} = 1 < \infty.$$

So the system $\boxed{\text{is BIBO stable.}}$

Exercise 2-10 A system's impulse response is $h(t) = u(t-1)/t$. Is the system BIBO stable?

Solution: The system is BIBO stable if and only if it is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)|\, dt$$

is finite. Here,

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| \frac{u(t-1)}{t} \right| dt = \int_{1}^{\infty} \frac{dt}{t} = \log(|t|)|_1^{\infty} \rightarrow \infty.$$

So the system $\boxed{\text{is not BIBO stable.}}$

Exercise 2-11 A system's impulse response is

$$h(t) = (3 + j4)e^{-(1-j2)t}\, u(t) + (3 - j4)e^{-(1+j2)t}\, u(t).$$

Is the system BIBO stable?

Solution: The impulse response has the form

$$h(t) = \sum_{i=1}^{N} C_i e^{\gamma_i t}\, u(t).$$

The system is BIBO stable if and only if all of the exponential coefficients γ_i in the impulse response have negative real parts. Here, the real parts of $-1 + j2$ and $-1 - j2$ are both -1, which is negative.

So the system $\boxed{\text{is BIBO stable.}}$

Exercise 2-12 $\cos(t) \rightarrow \boxed{h(t) = e^{-t}u(t)} \rightarrow ?$

Solution:

$$A\cos(\omega t + \phi) \rightarrow \boxed{\hat{\mathbf{H}}(\omega)} \rightarrow |\hat{\mathbf{H}}(\omega)| A\cos(\omega t + \phi + \theta)$$

where $\theta = \angle \hat{\mathbf{H}}(\omega)$. Here, $A = 1$, $\omega = 1$ rad/s, $\phi = 0$, and the frequency response function $\hat{\mathbf{H}}(\omega)$ is

$$\hat{\mathbf{H}}(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt = \frac{1}{j\omega + 1}\,.$$

At $\omega = 1$ rad/s,

$$\hat{\mathbf{H}}(1) = \frac{1}{j1 + 1} = \frac{1}{\sqrt{2}} e^{-j45^\circ}.$$

So the output is $\boxed{\frac{1}{\sqrt{2}} \cos(t - 45^\circ).}$

Exercise 2-13 $2\cos(t) \rightarrow$ SYSTEM $\rightarrow 2\cos(2t)+2$. Initial conditions are zero. Is this system LTI?

Solution: No. The response of an LTI system to a sinusoid at a given frequency is another sinusoid at that same frequency. This is a crucial property of LTI systems.

An LTI system cannot create a sinusoid at a frequency different from that of its input.

This system has created sinusoids at frequencies $\omega = 0$ and $\omega = 2$, so it **is not LTI.**

Exercise 2-14 $\cos(2t) \rightarrow$ SYSTEM $\rightarrow 0$. Can we say that the system is not LTI?

Solution: No. An LTI system can eliminate a sinusoid at a given frequency.

An LTI system cannot create a sinusoid at a frequency different from that of its input. For example, the system *could* be the LTI system

$$y(t) = \frac{d^2x}{dt^2} + 4x,$$

since if $x(t) = \cos(2t)$, then

$$y(t) = \frac{d^2x}{dt^2} + 4x = -4\cos(2t) + 4\cos(2t) = 0.$$

But we do not *know* this.

Exercise 2-15 Which damping condition is exhibited by $h(t)$ of

$$\frac{d^2y}{dt^2} + 5\,\frac{dy}{dt} + 4y(t) = 2\,\frac{dx}{dt}.$$

Solution: The general second-order LCCDE is

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2\,y(t) = b_1\frac{dx}{dt} + b_2\,x(t).$$

We read off $a_1 = 5$ and $a_2 = 4$. Note that $b_1 = 2$ and $b_2 = 0$ are irrelevant as far as the damping condition is concerned.

Then $\alpha = a_1/2 = 5/2 = 2.5\ \text{s}^{-1}$, $\omega_0 = \sqrt{a_2} = \sqrt{4} = 2$ rad/s, and $\xi = \alpha/\omega_0 = 2.5/2 = 1.25$.

Since $\xi = 1.25 > 1$, the system is $\boxed{\text{overdamped.}}$

Exercise 2-16 For what constant a_1 is

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + 9y(t) = 2 \frac{dx}{dt}$$

critically damped?

Solution: The general second-order LCCDE is

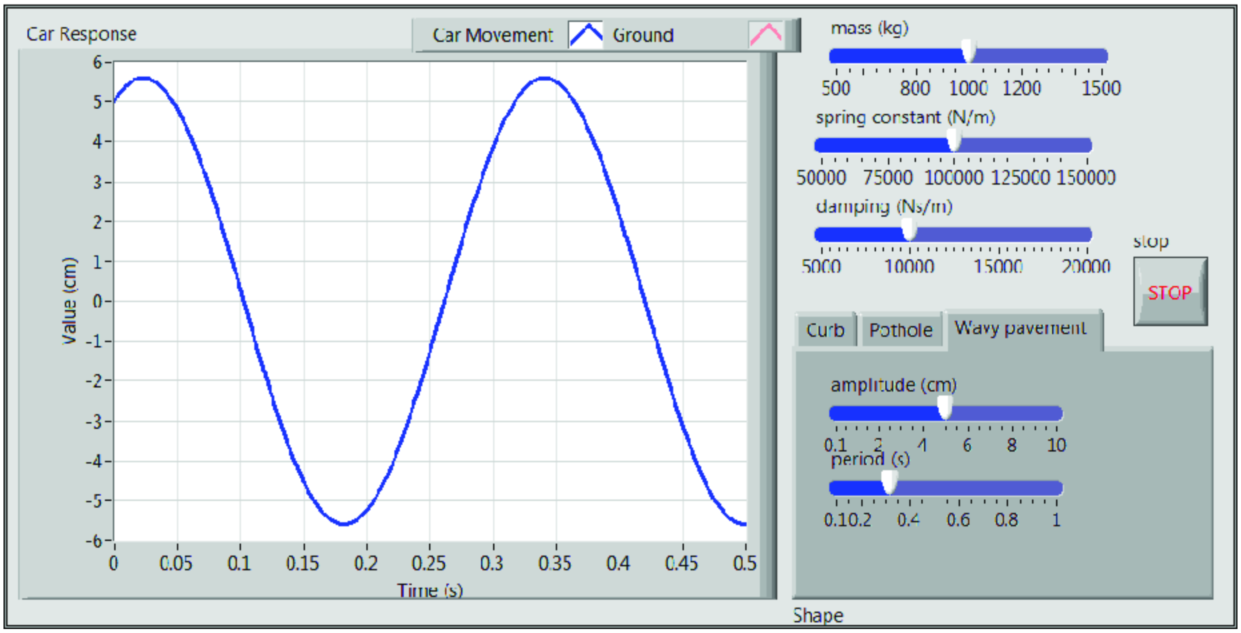
$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2\, y(t) = b_1 \frac{dx}{dt} + b_2\, x(t).$$

We read off $a_2 = 9$. Note that $b_1 = 2$ and $b_2 = 0$ are irrelevant here.
Then $\alpha = a_1/2$ and $\omega_0 = \sqrt{a_2} = \sqrt{4} = 2$ and $\xi = \alpha/\omega_0 = (a_1/2)/3 = a_1/6$.

The system is critically damped if $1 = \xi = a_1/6$, or $\boxed{a_1 = 6.}$

Exercise 2-17 Use LabVIEW Module 2.2 to compute the wavy pavement response in Example 2-19 and shown in Fig. 2-30.

Solution:



Exercise 3-1 Determine the Laplace transform of (a) $[\sin(\omega_0 t)]\, u(t)$, and (b) $r(t-T)$ [see ramp function in Chapter 1].

Solution:

(a) $[\sin\omega_0 t]\, u(t)$

$$\mathbf{X}(\mathbf{s}) = \int_{0^-}^{\infty} [\sin\omega_0 t]\, u(t)\, e^{-\mathbf{s}t}\, dt.$$

Application of the identity

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j},$$

$$\mathbf{X}(\mathbf{s}) = \frac{1}{2j}\int_0^{\infty} e^{j\omega_0 t} e^{-\mathbf{s}t}\, dt - \frac{1}{2j}\int_0^{\infty} e^{-j\omega_0 t} e^{-\mathbf{s}t}\, dt$$

$$= \frac{1}{2j}\left(\frac{e^{(j\omega_0 - \mathbf{s})t}}{j\omega_0 - \mathbf{s}} - \frac{e^{-(j\omega_0 + \mathbf{s})t}}{-(j\omega_0 + \mathbf{s})}\right)\Bigg|_0^{\infty}$$

$$= \frac{1}{2j}\left(\frac{-1}{j\omega_0 - \mathbf{s}} + \frac{-1}{j\omega_0 + \mathbf{s}}\right) = \boxed{\frac{\omega_0}{s^2 + \omega_0^2}}\,.$$

(b) $r(t-T) = (t-T)\, u(t-T)$

$$\mathbf{X}(\mathbf{s}) = \int_{0^-}^{\infty} (t-T)\, u(t-T)\, e^{-\mathbf{s}t}\, dt$$

$$= \int_T^{\infty} t e^{-\mathbf{s}t}\, dt - \int_T^{\infty} T e^{-\mathbf{s}t}\, dt.$$

Using the integral relation

$$\int x e^{ax}\, dx = \frac{e^{ax}}{a^2}\,(ax - 1),$$

we have

$$\mathbf{X}(\mathbf{s}) = \frac{e^{-\mathbf{s}t}}{\mathbf{s}^2}\,(-\mathbf{s}t - 1)\Bigg|_T^{\infty} + \frac{T}{\mathbf{s}}\, e^{-\mathbf{s}t}\Bigg|_T^{\infty}$$

$$= e^{-\mathbf{s}T}\left(\frac{\mathbf{s}T}{\mathbf{s}^2} + \frac{1}{\mathbf{s}^2} - \frac{T}{\mathbf{s}}\right) = \boxed{\frac{e^{-\mathbf{s}T}}{\mathbf{s}^2}}\,.$$

Exercise 3-2 Determine the Laplace transform of the causal sawtooth waveform shown in Fig. E3-2 (compare with Example 1-4).

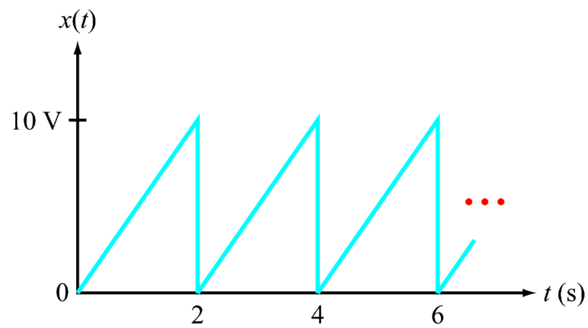


Figure E3-2

Solution: The sawtooth waveform is given by

$$x_1(t) = 5t[u(t) - u(t-2)] \text{ V}, \qquad \text{for cycle 1,}$$

$$x(t) = \sum_{n=0}^{\infty} 5(t-2n)[u(t-2n) - u(t-2(n+1))] \text{ V}, \qquad \text{for all cycles.}$$

Correspondingly,

$$\hat{\mathbf{X}}_1(\mathbf{s}) = \int_0^2 5te^{-\mathbf{s}t}\, dt = \frac{5}{\mathbf{s}^2}\,[1 - (2\mathbf{s}+1)e^{-2\mathbf{s}}],$$

$$\hat{\mathbf{X}}_2(\mathbf{s}) = \int_2^4 5(t-2)e^{-\mathbf{s}t}\, dt.$$

If we let $x = t - 2$,

$$\mathbf{X}_2(\mathbf{s}) = \int_0^2 5xe^{-2\mathbf{s}}e^{-\mathbf{s}x}\, dx$$
$$= e^{-2\mathbf{s}}\,\mathbf{X}_1(\mathbf{s}).$$

Similarly, for the mth cycle,

$$\mathbf{X}_m(\mathbf{s}) = e^{-m\mathbf{s}}\,\mathbf{X}_1(\mathbf{s}).$$

Hence,

$$\mathbf{X}(\mathbf{s}) = \mathbf{X}_1(\mathbf{s})\,(1 + e^{-2\mathbf{s}} + e^{-4\mathbf{s}} + e^{-6\mathbf{s}} + \cdots).$$

Using the series expansion

$$1 + x + x^2 + \cdots = \frac{1}{1-x},$$

we have

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{X}_1(\mathbf{s})}{1 - e^{-2\mathbf{s}}}$$
$$= \boxed{\frac{5[1 - (2\mathbf{s}+1)e^{-2\mathbf{s}}]}{s^2(1 - e^{-2\mathbf{s}})}}\,.$$

Exercise 3-3 Determine the poles and zeros of $\mathbf{X}(\mathbf{s}) = (\mathbf{s}+a)/[(\mathbf{s}+a)^2+\omega_0^2]$.

Solution: The zeros are the roots of the numerator polynomial set equal to zero.

$(\mathbf{s}+a) = 0 \Rightarrow$ $\boxed{\mathbf{z} = -a + j0.}$

The poles are the roots of the denominator polynomial set equal to zero.

$[(\mathbf{s}+a)^2+\omega_0^2] = 0 \Rightarrow$ $\boxed{\mathbf{p_1} = (-a - j\omega_0) \text{ and } \mathbf{p_2} = (-a + j\omega_0).}$

Exercise 3-4 Determine $\mathscr{L}\{[\sin\omega_0(t-T)]\,u(t-T)\}$.

Solution: According to Exercise 3-1(a),

$$[\sin(\omega_0 t)]\,u(t) \quad \leftrightarrow \quad \frac{\omega_0}{\mathbf{s}^2+\omega_0^2}\,.$$

Application of the shift property given by Eq. (3.16)

$$x(t-T)\,u(t-T) \quad \leftrightarrow \quad e^{-T\mathbf{s}}\,\mathbf{X}(\mathbf{s})$$

leads to

$$[\sin\omega_0(t-T)]\,u(t-T) \quad \leftrightarrow \quad \boxed{e^{-T\mathbf{s}}\,\frac{\omega_0}{\mathbf{s}^2+\omega_0^2}\,.}$$

Exercise 3-5 (a) Prove Eq. (3.20) and (b) apply it to determine $\mathscr{L}[e^{-at}\cos(\omega_0 t)\, u(t)]$.

Solution:

(a) If

$$x(t) \quad \longleftrightarrow \quad \mathbf{X}(\mathbf{s}),$$

then

$$\begin{aligned}\int_{0^-}^{\infty} e^{-at}\, x(t)\, e^{-\mathbf{s}t}\, dt &= \int_{0^-}^{\infty} x(t)\, e^{-(\mathbf{s}+a)t}\, dt \\ &= \int_{0^-}^{\infty} x(t)\, e^{-\mathbf{s}'t}\, dt \\ &= \mathbf{X}(\mathbf{s}') \\ &= \mathbf{X}(\mathbf{s}+a),\end{aligned}$$

where we temporarily used the substitution

$$\mathbf{s}' = \mathbf{s}+a.$$

Hence,

$$e^{-at}\, x(t) \quad \longleftrightarrow \quad \mathbf{X}(\mathbf{s}+a).$$

(b) Since

$$[\cos\omega_0 t]\, u(t) \quad \longleftrightarrow \quad \frac{\mathbf{s}}{\mathbf{s}^2+\omega_0^2}\,,$$

it follows that

$$[e^{-at}\cos(\omega_0 t)]\, u(t) \quad \longleftrightarrow \quad \frac{(\mathbf{s}+a)}{(\mathbf{s}+a)^2+\omega_0^2}\,.$$

Exercise 3-6 Determine the initial and final values of $x(t)$ if its Laplace transform is given by

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{s}^2 + 6\mathbf{s} + 18}{\mathbf{s}(\mathbf{s}+3)^2} \,.$$

Solution:

$$\begin{aligned} x(0^+) &= \lim_{\mathbf{s}\to\infty} \mathbf{s}\,\mathbf{X}(\mathbf{s}) \\ &= \lim_{\mathbf{s}\to\infty} \frac{\mathbf{s}^2 + 6\mathbf{s} + 18}{(\mathbf{s}+3)^2} = \boxed{1,} \\ x(\infty) &= \lim_{\mathbf{s}\to 0} \mathbf{s}\,\mathbf{X}(\mathbf{s}) \\ &= \lim_{\mathbf{s}\to\infty} \frac{\mathbf{s}^2 + 6\mathbf{s} + 18}{(\mathbf{s}+3)^2} = \boxed{2.} \end{aligned}$$

Exercise 3-7 Obtain the Laplace transform of: (a) $x_1(t) = 2(2 - e^{-t})\,u(t)$ and (b) $x_2(t) = e^{-3t}\cos(2t + 30^\circ)\,u(t)$.

Solution:

(a)

$$\begin{aligned} x_1(t) &= 2(2 - e^{-t})\,u(t) \\ &= (4 - 2e^{-t})\,u(t). \end{aligned}$$

By entries #2 and #3 in Table 3-2,

$$\mathbf{X}_1(\mathbf{s}) = \frac{4}{\mathbf{s}} - \frac{2}{\mathbf{s}+1} = \frac{4\mathbf{s}+4-2\mathbf{s}}{\mathbf{s}(\mathbf{s}+1)} = \frac{2\mathbf{s}+4}{\mathbf{s}(\mathbf{s}+1)}\,.$$

(b)

$$\begin{aligned} x_2(t) &= e^{-3t}\cos(2t+30^\circ)\,u(t) \\ &= e^{-3t}\,x_a(t), \end{aligned}$$

with

$$x_a(t) = \cos(2t + 20^\circ)\,u(t).$$

Applying entry #12 in Table 3-2 gives

$$\mathbf{X}_a(\mathbf{s}) = \frac{\mathbf{s}\cos 30^\circ - 2\sin 30^\circ}{\mathbf{s}^2+4} = \frac{0.866\mathbf{s}-1}{\mathbf{s}^2+4}\,.$$

Application of property #5 in Table 3-2 leads to

$$\begin{aligned} \mathbf{X}_2(\mathbf{s}) &= \mathbf{X}_a(\mathbf{s}+3) \\ &= \frac{0.866(\mathbf{s}+3)-1}{(\mathbf{s}+3)^2+4} \\ &= \boxed{\frac{0.866\mathbf{s}+1.6}{\mathbf{s}^2+6\mathbf{s}+13}\,.} \end{aligned}$$

Exercise 3-8 Apply the partial-fraction expansion method to determine $x(t)$, given that its Laplace transform is

$$\mathbf{X}(\mathbf{s}) = \frac{10\mathbf{s}+16}{\mathbf{s}(\mathbf{s}+2)(\mathbf{s}+4)} \,.$$

Solution: By partial-fraction expansion,

$$\mathbf{X}(\mathbf{s}) = \frac{A_1}{\mathbf{s}} + \frac{A_2}{\mathbf{s}+2} + \frac{A_3}{\mathbf{s}+4} \,,$$

with

$$\begin{aligned}
A_1 &= \mathbf{s}\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=0} \\
&= \left.\frac{10\mathbf{s}+16}{(\mathbf{s}+2)(\mathbf{s}+4)}\right|_{\mathbf{s}=0} = 2, \\
A_2 &= (\mathbf{s}+2)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-2} \\
&= \left.\frac{10\mathbf{s}+16}{\mathbf{s}(\mathbf{s}+4)}\right|_{\mathbf{s}=-2} = \frac{-20+16}{-2(2)} = 1, \\
A_3 &= (\mathbf{s}+4)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-4} \\
&= \left.\frac{10\mathbf{s}+16}{\mathbf{s}(\mathbf{s}+2)}\right|_{\mathbf{s}=-4} = \frac{-40+16}{-4(-2)} = -3.
\end{aligned}$$

Hence,

$$\mathbf{X}(\mathbf{s}) = \frac{2}{\mathbf{s}} + \frac{1}{\mathbf{s}+2} - \frac{3}{\mathbf{s}+4} \,,$$

and

$$x(t) = \boxed{[2+e^{-2t}-3e^{-4t}]\,u(t).}$$

Exercise 3-9 Determine the inverse Laplace transform of

$$\mathbf{X}(\mathbf{s}) = \frac{4\mathbf{s}^2 - 15\mathbf{s} - 10}{(\mathbf{s}+2)^3} \; .$$

Solution:

$$\mathbf{X}(\mathbf{s}) = \frac{4\mathbf{s}^2 - 15\mathbf{s} - 10}{(\mathbf{s}+2)^3} \; .$$

By partial-fraction expansion,

$$\mathbf{X}(\mathbf{s}) = \frac{B_1}{\mathbf{s}+2} + \frac{B_2}{(\mathbf{s}+2)^2} + \frac{B_3}{(\mathbf{s}+2)^3} \; ,$$

with

$$\begin{aligned}
B_3 &= (\mathbf{s}+2)^3 \, \mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-2} \\
&= 4\mathbf{s}^2 - 15\mathbf{s} - 10\big|_{\mathbf{s}=-2} = 16 + 30 - 10 = 36, \\
B_2 &= \frac{d}{d\mathbf{s}} \left[(\mathbf{s}+2)^3 \, \mathbf{X}(\mathbf{s})\right]\big|_{\mathbf{s}=-2} \\
&= \frac{d}{d\mathbf{s}} \left(4\mathbf{s}^2 - 15\mathbf{s} - 10\right)\big|_{\mathbf{s}=-2} = 8\mathbf{s} - 15|_{\mathbf{s}=-2} = -31, \\
B_1 &= \frac{1}{2} \, \frac{d}{d\mathbf{s}^2} \left(4\mathbf{s}^2 - 15\mathbf{s} - 10\right)\big|_{\mathbf{s}=-2} = 4.
\end{aligned}$$

Hence,

$$\mathbf{X}(\mathbf{s}) = \frac{4}{\mathbf{s}+2} - \frac{31}{(\mathbf{s}+2)^2} + \frac{36}{(\mathbf{s}+2)^3} \; .$$

By entries #3, #6, and #7 in Table 3-2,

$$x(t) = (4 - 31t + 18t^2)e^{-2t} \, u(t).$$

Exercise 3-10 Determine the inverse Laplace transform of

$$\mathbf{X}(\mathbf{s}) = \frac{2\mathbf{s}+14}{\mathbf{s}^2+6\mathbf{s}+25}\,.$$

Solution:

$$\begin{aligned}\mathbf{X}(\mathbf{s}) &= \frac{2\mathbf{s}+14}{\mathbf{s}^2+6\mathbf{s}+25}\\ &= \frac{2\mathbf{s}+14}{(\mathbf{s}+3-j4)(\mathbf{s}+3+j4)}\,.\end{aligned}$$

By partial fraction expansion

$$\mathbf{X}(\mathbf{s}) = \frac{\mathbf{B}_1}{\mathbf{s}+3-j4} + \frac{\mathbf{B}_1^*}{\mathbf{s}+3+j4}\,,$$

with

$$\begin{aligned}\mathbf{B}_1 &= (\mathbf{s}+3-j4)\,\mathbf{X}(\mathbf{s})\big|_{\mathbf{s}=-3+j4}\\ &= \left.\frac{(2\mathbf{s}+14)}{(\mathbf{s}+3+j4)}\right|_{\mathbf{s}=-3+j4}\\ &= \frac{-6+j8+14}{j8} = 1-j = \sqrt{2}\,e^{-j45^\circ}.\end{aligned}$$

Hence,

$$\mathbf{X}(\mathbf{s}) = \frac{\sqrt{2}\,e^{-j45^\circ}}{\mathbf{s}+3-j4} + \frac{\sqrt{2}\,e^{j45^\circ}}{\mathbf{s}+3+j4}\,.$$

By entry #15 in Table 3-2,

$$x(t) = \boxed{[2\sqrt{2}\,e^{-3t}\cos(4t-45^\circ)]\;u(t).}$$

Exercise 3-11 Is the system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{s}+1}{(\mathbf{s}+j3)(\mathbf{s}-j3)}$$

BIBO stable?

Solution: An LTI system is BIBO stable only if all of its poles are in the left half-plane.

The poles are the roots of the denominator polynomial set equal to zero.

$(\mathbf{s}+j3)(\mathbf{s}-j3) = 0$ ➡ $\mathbf{p} = \pm j3$. These poles are *on* the imaginary axis $\mathfrak{Re}[\mathbf{s}] = 0$, so they are *not* in the LHP, and the system is *not* BIBO stable.

In fact, the response to the (bounded) input $x(t) = \cos(3t)u(t)$ is the (unbounded) output

$$y(t) = 0.167\cos(3t-1.5708)\, u(t) + 0.527t\cos(3t-0.328)\, u(t),$$

which blows up as $t \to \infty$.

This can be derived as follows:

$$\mathbf{X}(\mathbf{s}) = \mathscr{L}[x(t)] = \mathscr{L}[\cos(3t)\, u(t)] = \frac{\mathbf{s}}{\mathbf{s}^2+3^2} = \frac{\mathbf{s}}{(\mathbf{s}+j3)(\mathbf{s}-j3)}\,.$$

Then

$$\mathbf{Y}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\,\mathbf{X}(\mathbf{s}) = \frac{\mathbf{s}+1}{(\mathbf{s}-j3)(\mathbf{s}+j3)}\,\frac{\mathbf{s}}{(\mathbf{s}-j3)(\mathbf{s}+j3)}$$

$$= \frac{\mathbf{s}^2+\mathbf{s}}{(\mathbf{s}-j3)^2(\mathbf{s}+j3)^2} = \frac{\mathbf{s}^2+\mathbf{s}}{\mathbf{s}^4+18\mathbf{s}^2+81}\,.$$

The partial fraction expansion of $\mathbf{Y}(\mathbf{s})$ is

$$\mathbf{Y}(\mathbf{s}) = \frac{\mathbf{A}}{\mathbf{s}-j3} + \frac{\mathbf{A}^*}{\mathbf{s}+j3} + \frac{\mathbf{B}}{(\mathbf{s}-j3)^2} + \frac{\mathbf{B}^*}{(\mathbf{s}+j3)^2}\,.$$

The residues $\mathbf{A}$ and $\mathbf{B}$ can be computed as follows:

$$\mathbf{B} = \frac{\mathbf{s}^2+\mathbf{s}}{(\mathbf{s}+j3)^2(\mathbf{s}-j3)^2}(\mathbf{s}-j3)^2|_{s=j3}$$

$$= \frac{(j3)^2+j3}{(j3+j3)^2} = \frac{-9+j3}{-36} = 0.25 - j0.0833,$$

$$\mathbf{A} = \frac{d}{ds}\left[\frac{\mathbf{s}^2+\mathbf{s}}{(\mathbf{s}+j3)^2(\mathbf{s}-j3)^2}(\mathbf{s}-j3)^2\right]|_{s=j3}$$

$$= \frac{(\mathbf{s}+j3)^2[2\mathbf{s}+1]-[\mathbf{s}^2+\mathbf{s}]2(\mathbf{s}+j3)}{(\mathbf{s}+j3)^4}|_{s=j3}$$

$$= \frac{(j6)^2[1+j6]-[-9+j3]2(j6)}{(j6)^4} = -j0.0833.$$

The residues $\mathbf{A}$ and $\mathbf{B}$ can also be computed using MATLAB or Mathscript: `[R P]=residue([1 1 0],[1 0 18 0 81])`. The output is

```
   -0.0000 - 0.0833i      -0.0000 + 3.0000i
R=  0.2500 - 0.0833i   P= -0.0000 + 3.0000i
   -0.0000 + 0.0833i      -0.0000 - 3.0000i
    0.2500 + 0.0833i      -0.0000 - 3.0000i
```

Inserting these values,

$$\mathbf{Y}(\mathbf{s}) = \frac{-j0.0833}{\mathbf{s}-j3} + \frac{0.25-j0.0833}{(\mathbf{s}-j3)^2} + \frac{j0.0833}{\mathbf{s}+j3} + \frac{0.25+j0.0833}{(\mathbf{s}+j3)^2}\ .$$

The inverse Laplace transform of $\mathbf{Y}(\mathbf{s})$ is

$$\begin{aligned} y(t) &= -j0.0833e^{j3t}\ u(t) + (0.25-j0.0833)te^{j3t}\ u(t) \\ &\quad + j0.0833e^{-j3t}\ u(t) + (0.25+j0.0833)te^{-j3t}\ u(t). \end{aligned}$$

This can be simplified using the formula

$$\mathbf{A}e^{\mathbf{p}t} + \mathbf{A}^* e^{-\mathbf{p}t} = 2|\mathbf{A}|e^{p_\mathrm{R}t}\cos(p_\mathrm{I}t+\theta),$$

where $\mathbf{p} = p_\mathrm{R} + jp_\mathrm{I}$ and $\mathbf{A} = |\mathbf{A}|e^{j\theta}$.

Here, we set

$$\begin{aligned} -j0.08333 &= 0.0833e^{-j1.5708}, \\ 0.25-j0.0833 &= 0.2635e^{-j0.3281}, \end{aligned}$$

and

$$\mathbf{p} = 0 + j3.$$

This gives

$$y(t) = \boxed{0.167\cos(3t-1.5708)\ u(t) + 0.527t\cos(3t-0.328)\ u(t).}$$

Exercise 3-12 Is the system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{(\mathbf{s}+1)(\mathbf{s}+2)(\mathbf{s}+3)}{(\mathbf{s}+4)(\mathbf{s}+5)}$$

BIBO stable?

Solution: An LTI system is BIBO stable only if its transfer function is proper or strictly proper:
The degree of the numerator is equal to or less than the degree of the denominator.
For $\mathbf{H}(\mathbf{s})$, the degree of the numerator is 3 and the degree of the denominator is 2.
So the transfer function is improper, and the system is *not* BIBO stable.
To illustrate, the response to the (bounded) input $x(t) = u(t)$ will include an (unbounded) impulse, since

$$\mathbf{Y}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\mathbf{X}(\mathbf{s}) = \frac{(\mathbf{s}+1)(\mathbf{s}+2)(\mathbf{s}+3)}{(\mathbf{s}+4)(\mathbf{s}+5)}\frac{1}{s} = 1 + \frac{0.3}{s} + \frac{1.5}{s+4} - \frac{4.8}{s+5}$$

using MATLAB or Mathscript as follows:

```
[R P K]=residue(poly([-1 -2 -3]'),poly([0 -4 -5]')).
```

Then

$$y(t) = \mathscr{L}^{-1}[\mathbf{H}(\mathbf{s})] = \delta(t) + 0.3\,u(t) + 1.5e^{-4t}\,u(t) - 4.8e^{-5t}\,u(t).$$

Even though all of the poles are in the left half-plane, the system is **not BIBO stable.**

Exercise 3-13 A system has the impulse response

$$h(t) = \delta(t) - 2e^{-3t}\, u(t).$$

Find its inverse system.

Solution:

$$\mathbf{H}(\mathbf{s}) = \mathscr{L}[h(t)] = 1 - \frac{2}{\mathbf{s}+3} = \frac{\mathbf{s}+3}{\mathbf{s}+3} - \frac{2}{\mathbf{s}+3} = \frac{\mathbf{s}+1}{\mathbf{s}+3},$$

$$\mathbf{G}(\mathbf{s}) = \frac{1}{\mathbf{H}(\mathbf{s})} = \frac{\mathbf{s}+3}{\mathbf{s}+1} = 1 + \frac{2}{\mathbf{s}+1}$$

$$\Rightarrow g(t) = \mathscr{L}^{-1}[\mathbf{G}(\mathbf{s})] = \delta(t) + 2e^{-t}\, u(t).$$

Exercise 3-14 An LTI system has impulse response $h(t) = 3e^{-t}u(t) - 2e^{-2t}u(t)$. Determine the LCCDE description.

Solution: First, compute the transfer function $\mathbf{H}(\mathbf{s})$:

$$\mathbf{H}(\mathbf{s}) = \mathscr{L}[h(t)] = \frac{3}{\mathbf{s}+1}\frac{\mathbf{s}+2}{\mathbf{s}+2} - \frac{2}{\mathbf{s}+2}\frac{\mathbf{s}+1}{\mathbf{s}+1} = \frac{\mathbf{s}+4}{(\mathbf{s}+1)(\mathbf{s}+2)}.$$

Next, set $\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})}$ and cross-multiply:

$$\frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = \frac{\mathbf{s}+4}{(\mathbf{s}+1)(\mathbf{s}+2)} \rightarrow \mathbf{Y}(\mathbf{s})(\mathbf{s}+1)(\mathbf{s}+2)$$
$$= \mathbf{X}(\mathbf{s})(\mathbf{s}+4) \rightarrow \mathbf{Y}(\mathbf{s})(\mathbf{s}^2+3\mathbf{s}+2) = \mathbf{X}(\mathbf{s})(\mathbf{s}+4).$$

Finally, take the inverse Laplace transform to yield:

$$\boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{dx}{dt} + 4x.}$$

Exercise 3-15 Compute the impulse response of the system described by the LCCDE

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 3x.$$

Solution: Read off the transfer function $\mathbf{H}(\mathbf{s})$ from the LCCDE coefficients and compute its partial fraction expansion:

$$\mathbf{H}(\mathbf{s}) = \frac{3}{\mathbf{s}^2 + 5\mathbf{s} + 4} = \frac{1}{\mathbf{s}+1} - \frac{1}{\mathbf{s}+4}\,.$$

Then

$$h(t) = \mathcal{L}^{-1}[\mathbf{H}(\mathbf{s})] = \boxed{e^{-t}\,u(t) - e^{-4t}\,u(t).}$$

Exercise 3-16 Compute the poles and modes of the system with LCCDE

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{dx}{dt} + 2x.$$

Solution: Taking the Laplace transform of the LCCDE gives

$$\mathbf{Y}(\mathbf{s})(\mathbf{s}^2 + 3\mathbf{s} + 2) = \mathbf{X}(\mathbf{s})(\mathbf{s} + 2).$$

The modes are the roots of the polynomial multiplying $\mathbf{Y}(\mathbf{s})$:
$\mathbf{s}^2 + 3\mathbf{s} + 2 = 0 \Longrightarrow \{-1, -2\}$ are the modes of the system.
The transfer function is

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{Y}(\mathbf{s})}{\mathbf{X}(\mathbf{s})} = \frac{\mathbf{s}+2}{\mathbf{s}^2 + 3\mathbf{s} + 2} = \frac{\mathbf{s}+2}{(\mathbf{s}+1)(\mathbf{s}+2)} = \frac{1}{\mathbf{s}+1}\ .$$

The poles are the roots of the denominator polynomial:
$\mathbf{s} + 1 = 0 \Longrightarrow \{-1\}$ is the pole of the system. Note $\{\text{poles}\} \subset \{\text{modes}\}$.

Exercise 3-17 Compute the zero-input response of $\frac{dy}{dt} + 2y = 3\frac{dx}{dt} + 4x$ with $y(0) = 5$.

Solution: For the zero-input response, set $x(t) = 0$. The LCCDE becomes $\frac{dy}{dt} + 2y = 0$.
This has the general solution $y(t) = Ce^{-2t}\, u(t)$ for some constant C.

$y(0) = 5 \Longrightarrow \boxed{y(t) = 5e^{-2t}u(t)}$ is the zero-input response.

Exercise 4-1 Convert the circuit in Fig. E4-1 into the **s**-domain.

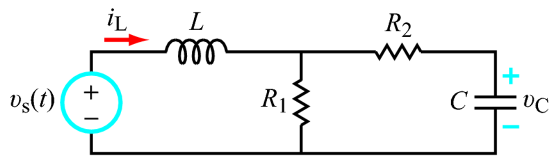
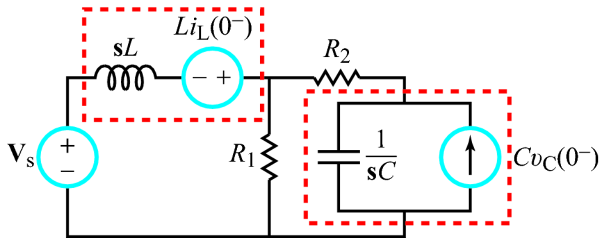


Figure E4-1

Solution:



Exercise 4-2 Compute the $\mathbf{s}$-domain impedance of a series RLC circuit with zero initial conditions. Simplify the expression to a ratio of polynomials.

Solution:

The $\mathbf{s}$-domain impedance of the resistor is $\mathbf{Z}_{\mathrm{R}}(\mathbf{s}) = R$.
The $\mathbf{s}$-domain impedance of the inductor is $\mathbf{Z}_{\mathrm{L}}(\mathbf{s}) = \mathbf{s}L$.
The $\mathbf{s}$-domain impedance of the capacitor is $\mathbf{Z}_{\mathrm{C}}(\mathbf{s}) = 1/(\mathbf{s}C)$.
The impedance of the series connection is the sum of the impedances:

$$\mathbf{Z}(\mathbf{s}) = \mathbf{Z_R}(\mathbf{s}) + \mathbf{Z_L}(\mathbf{s}) + \mathbf{Z_C}(\mathbf{s}) = R + \mathbf{s}L + 1/(\mathbf{s}C) = \boxed{\frac{\mathbf{s}^2 + \frac{R}{L}\mathbf{s} + \frac{1}{LC}}{\mathbf{s}/L}}.$$

Exercise 4-3 A mass is connected by a spring to a moving surface. What is its electrical analog?

Solution: The mass becomes a capacitor, the spring an inductor, and the surface a voltage source.

So the electrical analog is a series LC circuit driven by a voltage source, the same as the circuit in Fig. 4-7(b), but without the resistor.

Exercise 4-4 What do you expect the impulse response of the system in Exercise 4-3 to be like?

Solution: The impulse response of an LC circuit is

$$\boxed{h(t) = A\cos\left(\frac{t}{\sqrt{LC}} + \theta\right) u(t)}$$

for some A and θ.

This is a pure oscillation without damping (there is no resistor).

Exercise 4-5 In the SMD system shown in Fig. E4-5, $v_x(t)$ is the input velocity of the platform and $v_y(t)$ is the output velocity of mass m. Draw the equivalent **s**-domain circuit.

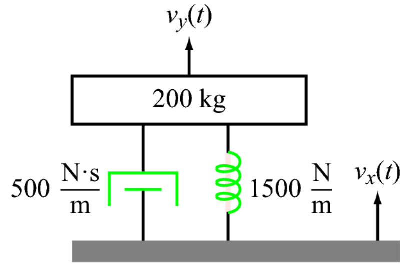
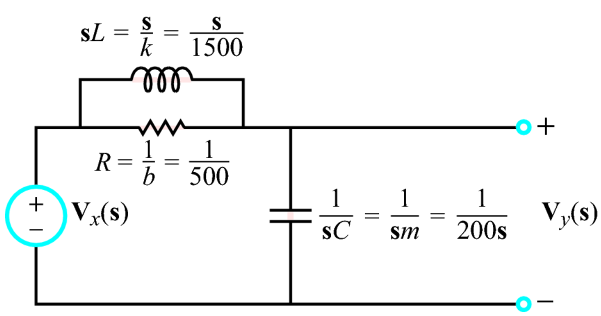


Figure E4-5

Solution:



Exercise 4-6 In the SMD system shown in Fig. E4-6, $v_x(t)$ is the input velocity of the platform and $v_y(t)$ is the output velocity of mass m. Draw the equivalent **s**-domain circuit.

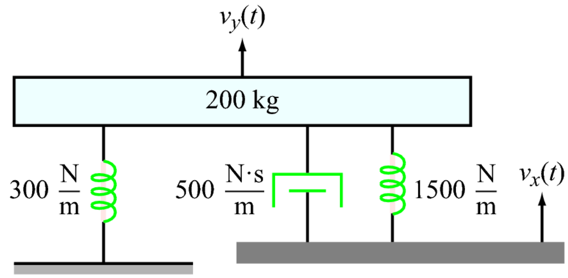
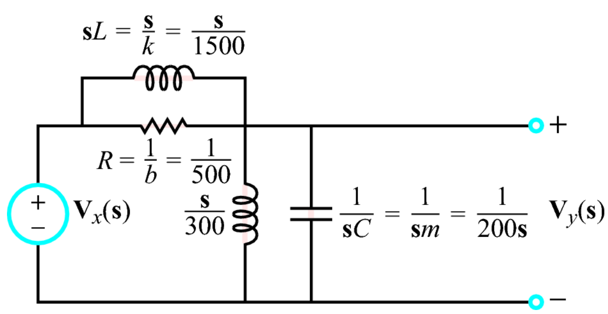


Figure E4-6

Solution:



Exercise 4-7 What is the amplitude of the head *displacement* for the person in Example 4-7, if the seat *displacement* is $x_1(t) = 0.02\cos(10t)$ (m)?

Solution:

$$\begin{aligned}
\upsilon_1(t) &= dx_1/dt \\
&= -0.2\sin(10t) = 0.2\cos(10t + 90^\circ) \text{ (m/s)}; \\
\upsilon_4(t) &= 0.2 \times 4.12\cos(10t + 90^\circ - 116.1^\circ) \\
&= 0.824\cos(10t - 26.1^\circ) \text{ (m/s)}; \\
x_4(t) &= \int_{-\infty}^{t} \upsilon_4(\tau)\, d\tau = 0.0824\sin(10t - 26.1^\circ) \text{ (m)}; \\
\text{amplitude} &= 8.24 \text{ cm.}
\end{aligned}$$

Exercise 4-8 Obtain the transfer function of the op-amp circuit shown in Fig. E4-8.

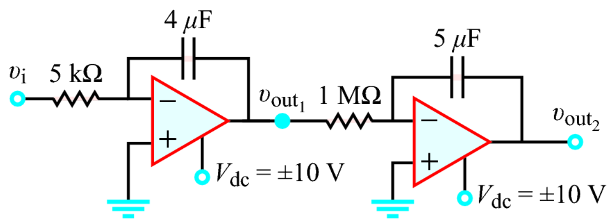


Figure E4-8

Solution: The circuit consists of two integrators connected in series.
The transfer function of an integrator op-amp circuit is

$$\mathbf{H}(\mathbf{s}) = \frac{-1}{\mathbf{s}RC} .$$

So

$$\mathbf{H}(\mathbf{s}) = \frac{-1}{(4\ \mu\text{F})(5\ \text{k}\Omega)\mathbf{s}} \frac{-1}{(5\ \mu\text{F})(1\ \text{M}\Omega)\mathbf{s}} = \frac{10}{\mathbf{s}^2} .$$

Exercise 4-9 How many op amps are needed, as a minimum, to implement a system with transfer function $\mathbf{H}(\mathbf{s})=\frac{b}{\mathbf{s}+a}$, where $a, b > 0$?

Solution: Use the one-pole configuration in Table 4-3 with

$$a = \frac{1}{R_\text{f} C_\text{f}}$$

and

$$b = \frac{1}{R_\text{i} C_\text{f}} .$$

Given any value of C_f, set

$$R_\text{i} = \frac{1}{b C_\text{f}}$$

and

$$R_\text{f} = \frac{1}{a C_\text{f}} .$$

But this implements $\frac{-b}{\mathbf{s}+a}$, not $\frac{b}{\mathbf{s}+a}$.

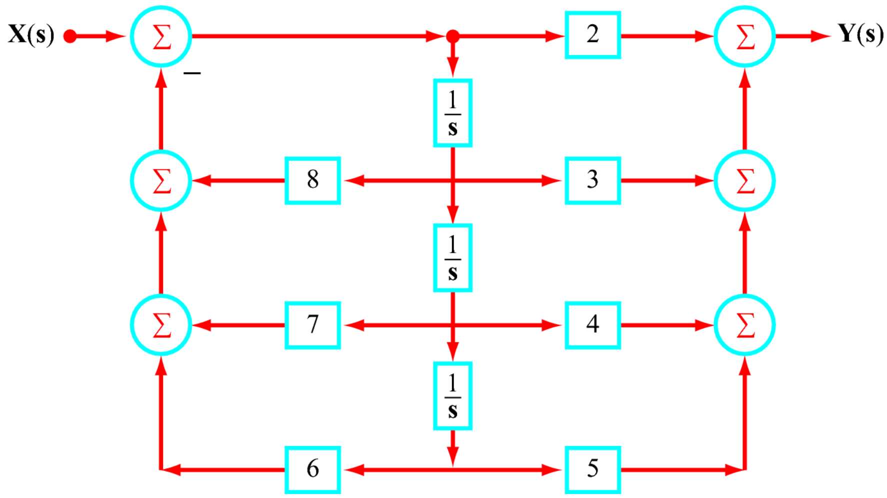
So we need a second op amp with a gain of (-1) to implement an inverter to get $\frac{b}{\mathbf{s}+a}$.

Hence, we need $\boxed{\text{2 op amps}}$.

Exercise 4-10 Using Direct Form II, determine how many integrators are needed to realize the system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{2\mathbf{s}^3 + 3\mathbf{s}^2 + 4\mathbf{s} + 5}{\mathbf{s}^3 + 8\mathbf{s}^2 + 7\mathbf{s} + 6} \,.$$

Solution:



Exercise 4-11 What is the minimum value of the feedback factor K needed to stabilize a system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{1}{(\mathbf{s}+3)(\mathbf{s}-2)} \text{ ?}$$

Solution:

$$\mathbf{H}(\mathbf{s}) = \frac{1}{(\mathbf{s}+3)(\mathbf{s}-2)} = \frac{1}{\mathbf{s}^2+\mathbf{s}-6} .$$

The closed-loop transfer function is

$$\mathbf{Q}(\mathbf{s}) = \frac{\mathbf{H}(\mathbf{s})}{1+K\mathbf{H}(\mathbf{s})} = \frac{1/(\mathbf{s}^2+\mathbf{s}-6)}{1+K/(\mathbf{s}^2+\mathbf{s}-6)} = \frac{1}{\mathbf{s}^2+\mathbf{s}-6+K} .$$

A quadratic polynomial has both of its roots in the open left half-plane (OLHP) if and only if all three coefficients have the same sign. So we need $\boxed{K > 6.}$

Exercise 4-12 What values of K can stabilize a system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{1}{(\mathbf{s}-3)(\mathbf{s}+2)} \text{ ?}$$

Solution:

$$\mathbf{H}(\mathbf{s}) = \frac{1}{(\mathbf{s}-3)(\mathbf{s}+2)} = \frac{1}{\mathbf{s}^2-\mathbf{s}-6}\,.$$

The closed-loop transfer function is

$$\mathbf{Q}(\mathbf{s}) = \frac{\mathbf{H}(\mathbf{s})}{1+K\mathbf{H}(\mathbf{s})} = \frac{1/(\mathbf{s}^2-\mathbf{s}-6)}{1+K/(\mathbf{s}^2-\mathbf{s}-6)} = \frac{1}{\mathbf{s}^2-\mathbf{s}-6+K}\,.$$

A quadratic polynomial has both of its roots in the open left half-plane (OLHP) if and only if all three coefficients have the same sign.

The quadratic and linear term coefficients have opposite signs: 1 and -1.

So at least one of its roots is not in the LHP, and the system is unstable.

No value of K can stabilize the system.

Exercise 4-13 What is the time constant of an oven whose heat capacity is 20 J/°C and thermal resistance is 5°C/W?

Solution: The time constant is

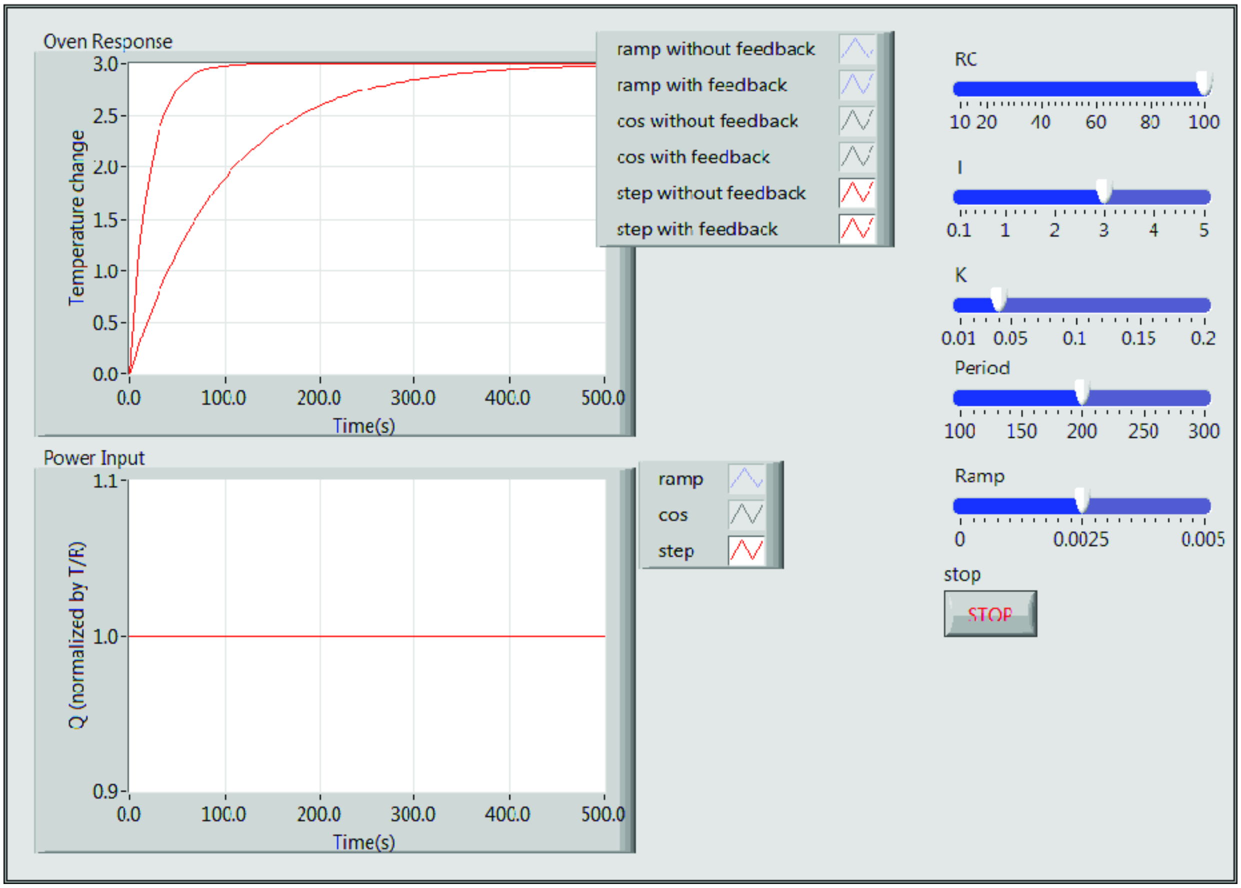
$$\tau_c = \frac{1}{a} = RC = (20\ \text{J}/^\circ\text{C})(5^\circ\text{C/W}) = 100\ \text{J/W} = \boxed{100\ \text{s.}}$$

Exercise 4-14 What is the closed-loop time constant when feedback with $K = 0.04\ \text{s}^{-1}$ is used on the oven of Exercise 4-13?

Solution: Using Eq. (4.122b), $b = a + K = \frac{1}{100} + 0.04 = 0.05$. The closed-loop time constant is $\boxed{20\ \text{s.}}$

Exercise 4-15 Use LabVIEW Module 4.1 to compute the oven temperature responses shown in Fig. 4-32, using values given in the text.

Solution:



Exercise 4-16 In open-loop mode, an op-amp circuit has a gain of 100 dB and half-power bandwidth of 32 Hz. What will the gain and bandwidth be in closed-loop mode with $K = 0.01$?

Solution: Converting units, 100 dB $= 10^5$ and 32 Hz corresponds to $\omega = 2\pi 32 = 200$ rad/s.

So the gain-bandwidth product of the amplifier is GBP $= 2 \times 10^7$.

The closed-loop dc gain is $\frac{1}{K} = 100$.

The closed-loop half-power bandwidth is

$$\frac{2 \times 10^7}{100} = 2 \times 10^5 \text{ rad/s},$$

which is equivalent to $\boxed{\text{32 kHz.}}$ The closed-loop gain is $2 \times 10^7 / 2 \times 10^5 = 100$, which is equivalent to $\boxed{\text{40 dB.}}$

Exercise 4-17 Compute the steady-state step response $\lim_{t\to\infty} y_{\text{step}}(t)$ for the BIBO stable system with transfer function

$$\mathbf{H}(\mathbf{s}) = \frac{2\mathbf{s}^2 + 3\mathbf{s} + 4}{5\mathbf{s}^3 + 6\mathbf{s}^2 + 7\mathbf{s} + 8} .$$

Solution: We can avoid computing a partial fraction expansion by using the Final Value Theorem.

Since $\mathscr{L}[u(t)] = 1/\mathbf{s}$, we have

$$\mathscr{L}[y_{\text{step}}(t)] = \mathbf{Y}_{\text{step}}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\,\frac{1}{\mathbf{s}} .$$

The Final Value Theorem gives

$$\lim_{t\to\infty} y_{\text{step}}(t) = \lim_{\mathbf{s}\to 0} \mathbf{s}\,\mathbf{Y}_{\text{step}}(\mathbf{s}) = \lim_{\mathbf{s}\to 0} \mathbf{s}\,\mathbf{H}(\mathbf{s})\,\frac{1}{\mathbf{s}} = \mathbf{H}(0) = \frac{4}{8} = \boxed{\frac{1}{2}} .$$

Exercise 4-18 In Example 4-13, suppose $a = 101$, $b = 100$ and $K_1 = 1$. Compute K_2 so that the closed-loop system is critically damped using PD feedback.

Solution: The closed-loop transfer function for the motor is given in Eq. (4.148) for proportional feedback, $\mathbf{G}(\mathbf{s}) = K$. If PD feedback is used instead, Eq. (4.148) should be modified to

$$\mathbf{Q}(\mathbf{s}) = \frac{b}{\mathbf{s}^2 + a\mathbf{s} + b\,\mathbf{G}(\mathbf{s})},$$

with $\mathbf{G}(\mathbf{s}) = K_1 + K_2\mathbf{s}$.

Inserting the given values,

$$\mathbf{Q}(\mathbf{s}) = \frac{100}{\mathbf{s}^2 + 101\mathbf{s} + 100(1 + K_2\mathbf{s})} = \frac{100}{\mathbf{s}^2 + (101 + 100K_2)\mathbf{s} + 100}\,.$$

The closed-loop system is critically damped if the denominator polynomial

$$[\mathbf{s}^2 + (101 + 100K_2)\mathbf{s} + 100]$$

has a double root. This happens if

$$\mathbf{s}^2 + (101 + 100K_2)\mathbf{s} + 100 = (s + 10)^2 \Longrightarrow (101 + 100K_2) = 20 \Longrightarrow K_2 = -0.81.$$

The impulse response is then $h(t) = \mathscr{L}^{-1}[\mathbf{Q}(\mathbf{s})] = \boxed{100te^{-10t}\,u(t).}$

Exercise 4-19 Using proportional feedback with $K = L + 0.2$, compute the response to input $x(t) = 0.01u(t)$.

Solution: From Eq. (4.170), the closed-loop transfer function is

$$\mathbf{Q}(\mathbf{s}) = \frac{-\mathbf{s}^2}{(L-K)\mathbf{s}^2 - g} = \frac{-\mathbf{s}^2}{-0.2\mathbf{s}^2 - 9.8} = \frac{5\mathbf{s}^2}{\mathbf{s}^2 + 49}.$$

Then

$$\theta(\mathbf{s}) = \mathbf{Q}(\mathbf{s})\,\mathbf{X}(\mathbf{s}) = \frac{5\mathbf{s}^2}{\mathbf{s}^2+49}\,\frac{0.01}{\mathbf{s}} = \frac{0.05\mathbf{s}}{\mathbf{s}^2+49},$$

and

$$\theta(t) = \mathscr{L}^{-1}[\theta(\mathbf{s})] = \boxed{0.05\cos(7t)\,u(t),}$$

which is oscillatory.

Note the amplitude 0.05 is small enough for the linear model to be valid.

Exercise 4-20 Using PI feedback, show that the closed-loop system is stable if $K_1 > L$ and $K_2 > 0$.

Solution: PI feedback means that $\mathbf{G(s)} = K_1 + K_2/\mathbf{s}$. The closed-loop transfer function is given by Eq. (4.177) as

$$\mathbf{Q(s)} = \frac{-\mathbf{s}^2}{(L - K_1)\mathbf{s}^2 - K_2\mathbf{s} - g}\ .$$

The closed-loop system is BIBO stable if all three coefficients of the denominator polynomial have the same sign, so its roots are in the LHP. If $K_1 > L$ and $K_2 > 0$, then all three coefficients are negative, and the closed-loop system **is BIBO stable.**

Exercise 4-21 Using PI feedback with $K_1 = L + 0.2$, select the value of K_2 so that the closed-loop system is critically damped.

Solution: PI feedback means that $\mathbf{G}(\mathbf{s}) = K_1 + K_2/\mathbf{s}$. From Eq. (4.177), the closed-loop transfer function is

$$\mathbf{Q}(\mathbf{s}) = \frac{-\mathbf{s}^2}{(L-K_1)\mathbf{s}^2 - K_2\mathbf{s} - g}.$$

Inserting $K_1 = L + 0.2$ gives

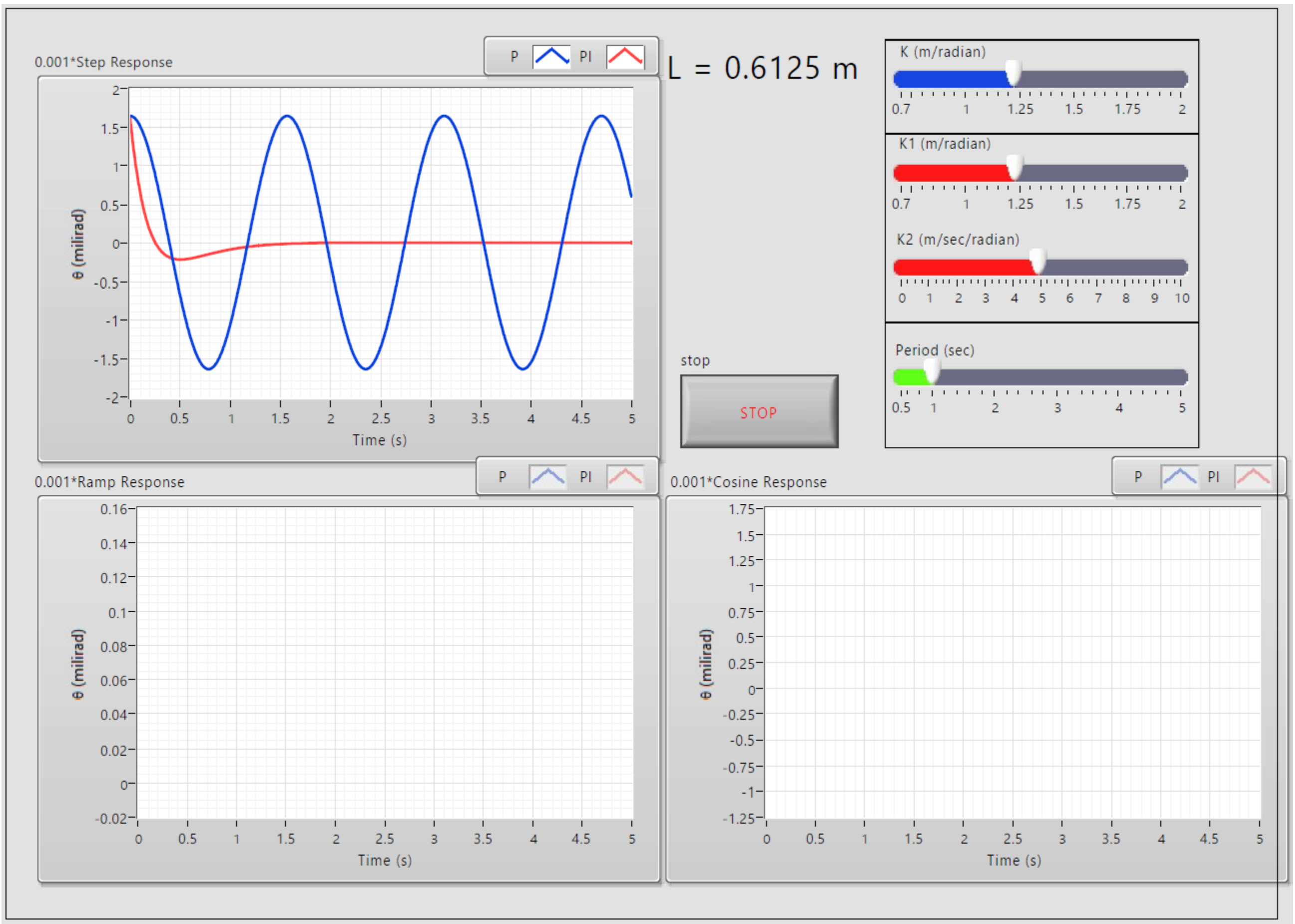
$$\mathbf{Q}(\mathbf{s}) = \frac{-\mathbf{s}^2}{-0.2\mathbf{s}^2 - K_2\mathbf{s} - 9.8}\,\frac{-5}{-5} = \frac{5\mathbf{s}^2}{\mathbf{s}^2 + 5K_2\mathbf{s} + 49}.$$

The closed-loop system is critically damped if it has a double pole.

This happens if $\mathbf{s}^2 + 5K_2\mathbf{s} + 49 = (\mathbf{s}+7)^2 \Longrightarrow 5K_2 = 14 \Longrightarrow \boxed{K_2 = 2.8.}$

Exercise 4-22 Use LabVIEW Module 4.2 to compute the inverted pendulum responses shown in Fig. 4-38.

Solution:



Exercise 5-1 Obtain the Fourier-series representation for the waveform shown in Fig. E5-1.

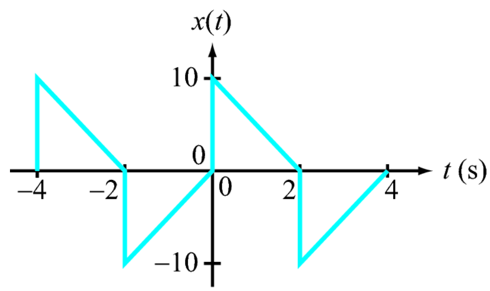


Figure E5-1

Solution: For the cycle from $t = -2$ s to $t = 2$ s, the waveform is given by

$$x(t) = \begin{cases} 5t & \text{for } -2 \le t \le 0, \\ 10 - 5t & \text{for } 0 \le t \le 2. \end{cases}$$

With $T_0 = 4$ s and $\omega_0 = 2\pi/T_0 = \pi/2$ rad/s,

$$a_0 = \frac{1}{T_0}\int_{-2}^{2} x(t)\, dt$$
$$= \frac{1}{4}\left[\int_{-2}^{0} 5t\, dt + \int_{0}^{2} (10 - 5t)\, dt\right] = 0,$$
$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos n\omega_0 t\, dt$$
$$= \frac{1}{2}\left[\int_{-2}^{0} 5t\cos\frac{n\pi t}{2}\, dt + \int_{0}^{2} (10 - 5t)\cos\frac{n\pi t}{2}\, dt\right].$$

Using the integral relationship given in Appendix C-2 as

$$\int x\cos ax\, dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax,$$

we have

$$a_n = \frac{20}{n^2\pi^2}\,(1 - \cos n\pi).$$

Similarly, using the relation

$$\int x\sin ax\, dx = \frac{1}{a^2}\sin ax = \frac{x}{a}\cos ax,$$

we have

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\ \sin n\omega_0 t\, dt$$
$$= \frac{1}{2}\left[\int_{-2}^{0} 5t\sin\frac{n\pi t}{2}\, dt + \int_{0}^{2} (10 - 5t)\sin\frac{n\pi t}{2}\, dt\right]$$
$$= \frac{10}{n\pi}\,(1 - \cos n\pi).$$

Hence,

$$x(t) = \sum_{n=1}^{\infty}\left[\frac{20}{n^2\pi^2}\,(1 - \cos n\pi)\cos\frac{n\pi t}{2} + \frac{10}{n\pi}\,(1 - \cos n\pi)\sin\frac{n\pi t}{2}\right].$$

Exercise 5-2 Obtain the line spectra associated with the periodic function of Exercise 5-1.

Solution:

$$\begin{aligned}
c_n &= \sqrt{a_n^2 + b_n^2} \\
&= \left\{ \left[\frac{20}{n^2\pi^2} (1 - \cos n\pi) \right]^2 + \left[\frac{10}{n\pi} (1 - \cos n\pi) \right]^2 \right\}^{1/2} \\
&= (1 - \cos n\pi) \frac{20}{n^2\pi^2} \sqrt{1 + \frac{n^2\pi^2}{4}}\,, \\
\phi_n &= -\tan^{-1}\left(\frac{b_n}{a_n}\right) \\
&= -\tan^{-1}\left(\frac{n\pi}{2}\right).
\end{aligned}$$

We note that $c_n = 0$ when $n =$ even.

Exercise 5-3 A periodic signal $x(t)$ has the complex exponential Fourier series

$$x(t) = (-2 + j0) + (3 + j4)e^{j2t} + (1 + j)e^{j4t} + (3 - j4)e^{-j2t} + (1 - j)e^{-j4t}.$$

Compute its cosine/sine and amplitude/phase Fourier series representations.

Solution: Using the relations in Table 5-3, we can assemble the following table:

n	$n\omega_0$	$\mathbf{x_n}$	$\mathbf{x_n}$	$c_n = 2\|\mathbf{x}_n\|$	$\phi_n = \angle \mathbf{x}_n$	$a_n = c_n \cos\phi_n$	$b_n = -c_n \sin\phi_n$
0	0	-2	$2e^{j180°}$	2	180°	-2	0
1	2	$3 + j4$	$5e^{j53°}$	10	53°	6	-8
2	4	$1 + j$	$\sqrt{2}e^{j45°}$	$2\sqrt{2}$	45°	2	-2

Amplitude/phase representation:

$$x(t) = -2 + 10\cos(2t + 53°) + 2\sqrt{2}\cos(4t + 45°).$$

Cosine/sine representation:

$$x(t) = -2 + 6\cos(2t) - 8\sin(2t) + 2\cos(4t) - 2\sin(4t).$$

Note that $\mathbf{x}_n = \frac{1}{2}(a_n - jb_n)$ for $n = 1$ and 2 but $|\mathbf{x}_0| = c_0$, not $c_0/2$.

Exercise 5-4 (a) Does the waveform $x(t)$ shown in Fig. E5-4 exhibit either even or odd symmetry? (b) What is the value of a_0? (c) Does the function $y(t) = x(t) - a_0$ exhibit either even or odd symmetry?

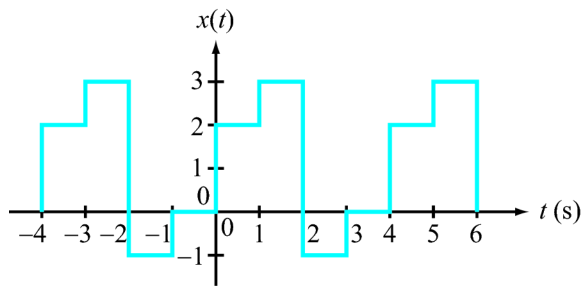


Figure E5-4

Solution:

(a)

$x(t) \neq x(-t)$ → no even symmetry

$x(t) \neq -x(-t)$ → no odd symmetry

(b)

$$a_0 = \frac{2 \times 1 + 3 \times 1 + (-1) \times 1}{4} = 1.$$

(c) $y(t) = [x(t) - a_0]$ has odd symmetry.

Exercise 5-5 The RL circuit shown in Fig. E5-5(a) is excited by the square-wave voltage waveform of Fig. E5-5(b). Determine $\upsilon_{\text{out}}(t)$.

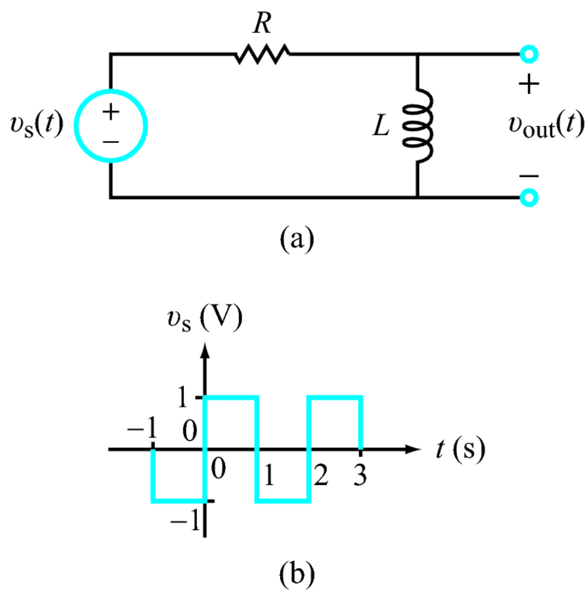


Figure E5-5

Solution: From the waveform, we deduce that

$$T_0 = 2 \text{ s}, \qquad \omega_0 = \frac{2\pi}{T_0} = \pi \text{ rad/s}, \qquad A = 1 \text{ V}.$$

Step 1:

From entry #2 in Table 5-4,

$$\begin{aligned}
\upsilon_s(t) &= \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{2\pi nt}{T_0}\right) \\
&= \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{4}{n\pi} \sin n\pi t \\
&= \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{4}{n\pi} \cos(n\pi t - 90^\circ) \text{ V}.
\end{aligned}$$

Thus,

$$c_0 = 0, \qquad c_n = \frac{4}{n\pi}, \qquad \phi_n = -90^\circ.$$

Step 2:

$$\hat{\mathbf{H}}(\omega) = \frac{\hat{\mathbf{V}}_{\text{out}}(\omega)}{\hat{\mathbf{V}}_s(\omega)} = \frac{j\omega L}{R + j\omega L}.$$

Step 3:

With $\omega_0 = \pi$ rad/s and $\phi_n = -90^\circ$,

$$\begin{aligned}
\upsilon_{\text{out}}(t) &= c_0\, \hat{\mathbf{H}}(\omega = 0) + \sum_{n=1}^{\infty} c_n\, \mathfrak{Re}\{\mathbf{H}(\omega = n\omega_0)\, e^{j(n\omega_0 t + \phi_n)}\} \\
&= \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4}{n\pi}\, \mathfrak{Re}\left\{ \frac{jn\omega_0 L}{R + jn\omega_0 L}\, e^{j(n\omega_0 t + \phi_n)} \right\} \\
&= \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4L}{\sqrt{R^2 + n^2\pi^2 L^2}} \cos(n\pi t + \theta_n) \;\; \text{V},
\end{aligned}$$

with

$$\theta_n = -\tan^{-1}\left(\frac{n\pi L}{R}\right).$$

Exercise 5-6 For a single rectangular pulse of width τ, what is the spacing $\Delta\omega$ between first nulls? If τ is very wide, will its frequency spectrum be narrow and peaked or wide and gentle?

Solution: From Fig. 5-13(b), first nulls occur at $\pm\frac{2\pi}{\tau}$. Hence, $\boxed{\Delta\omega = 4\pi/\tau.}$ Wide τ leads to narrow spectrum.

Exercise 5-7 Use the entries in Table 5-6 to determine the Fourier transform of $u(-t)$.

Solution: From Table 5-6,

$$\mathrm{sgn}(t) \quad \leftrightarrow \quad \frac{2}{j\omega},$$

$$u(t) \quad \leftrightarrow \quad \pi\,\delta(\omega) + \frac{1}{j\omega}.$$

Also,

$$\mathrm{sgn}(t) = u(t) - u(-t).$$

Hence,

$$u(-t) = u(t) - \mathrm{sgn}(t),$$

and the corresponding Fourier transform is

$$u(-t) \quad \leftrightarrow \quad \pi\,\delta(\omega) + \frac{1}{j\omega} - \frac{2}{j\omega} = \boxed{\pi\,\delta(\omega) - \frac{1}{j\omega}.}$$

Exercise 5-8 Verify the Fourier transform expression for entry #10 in Table 5-7.

Solution:

$$x(t)\cos(\omega_0 t) = \left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) x(t).$$

Applying Property 5 in Table 5-7,

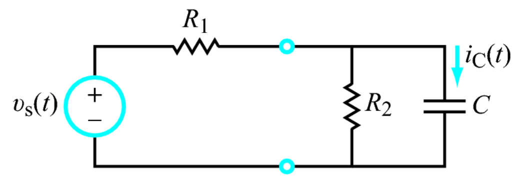
$$\frac{1}{2}\, e^{j\omega_0 t}\, x(t) \quad \leftrightarrow \quad \frac{1}{2}\, \mathbf{X}(\omega - \omega_0),$$

$$\frac{1}{2}\, e^{-j\omega_0 t}\, x(t) \quad \leftrightarrow \quad \frac{1}{2}\, \mathbf{X}(\omega + \omega_0).$$

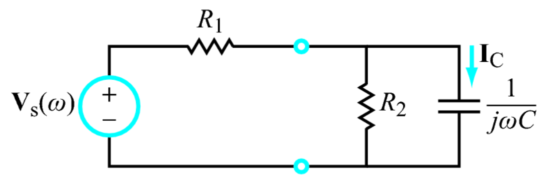
Hence,

$$x(t)\cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2}\, [\mathbf{X}(\omega - \omega_0) + \mathbf{X}(\omega + \omega_0)].$$

Exercise 5-9 Determine the voltage across the capacitor, $\upsilon_C(t)$, in Fig. 5-20(a) of Example 5-15, for each of the three voltage waveforms given in the example statement.



(a) Time domain



(b) ω-domain

From Eq. (5.120),

$$\frac{\hat{\mathbf{I}}_C(\omega)}{\hat{\mathbf{V}}_s(\omega)} = \frac{j0.5\omega \times 10^{-3}}{3 + j\omega}.$$

Hence, with $C = 0.25$ mF,

$$\hat{\mathbf{V}}_C(\omega) = \frac{\hat{\mathbf{I}}_C(\omega)}{j\omega C} = \frac{2}{3 + j\omega}\,\hat{\mathbf{V}}_s(\omega).$$

(a) $\upsilon_s(t) = 10u(t)$

$$\hat{\mathbf{V}}_s(\omega) = 10\pi\,\delta(\omega) + \frac{10}{j\omega}.$$

Hence,

$$\hat{\mathbf{V}}_C(\omega) = \frac{20\pi\,\delta(\omega)}{3 + j\omega} + \frac{20}{j\omega(3 + j\omega)},$$

$$\begin{aligned}\upsilon_C(t) = \mathcal{F}^{-1}[\hat{\mathbf{V}}_C(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{20\pi\,\delta(\omega)}{3 + j\omega}\,e^{j\omega t}\,d\omega + \mathcal{F}^{-1}\left[\frac{20}{j\omega(3 + j\omega)}\right] \\ &= \frac{10}{3}\,u(t) + \mathcal{F}^{-1}\left[\frac{20}{j\omega(3 + j\omega)}\right].\end{aligned}$$

From entry #7 in Table 5-6,

$$e^{-at}\,u(t) \quad \leftrightarrow \quad \frac{1}{a + j\omega}.$$

Let us define

$$\frac{20}{j\omega(3 + j\omega)} = \frac{\hat{\mathbf{F}}_1}{j\omega}.$$

with

$$\hat{\mathbf{F}}_1 = \frac{20}{3 + j\omega}.$$

Hence,

$$f_1(t) = 20e^{-3t}\,u(t).$$

According to property #8 in Table 5-7,

$$\int_{-\infty}^{t} f_1(t)\,dt \quad \leftrightarrow \quad \frac{\hat{\mathbf{F}}_1(\omega)}{j\omega}\,.$$

Hence,

$$\mathfrak{F}^{-1}\left[\frac{20}{j\omega(3+j\omega)}\right] = \int_{-\infty}^{t} 20e^{-3t}\,u(t)\,dt$$

$$= \frac{-20}{3}\,e^{-3t}\Big|_0^t = \frac{20}{3}\,(1-e)^{-3t})\,u(t).$$

Thus,

$$\upsilon_C(t) = \boxed{\left[\frac{10}{3} + \frac{20}{3}\,(1-e^{-3t})\right]\,u(t)\ \text{ V.}}$$

(b) $\upsilon_s(t) = 10e^{-2t}\,u(t)$ V.

$$\hat{\mathbf{V}}_s(\omega) = \frac{10}{2+j\omega}\ \text{V},$$

and

$$\hat{\mathbf{V}}_C(\omega) = \frac{2}{3+j\omega}\cdot\frac{10}{2+j\omega} = \frac{20}{(3+j\omega)(2+j\omega)}\,.$$

By partial fraction expansion,

$$\hat{\mathbf{V}}_C(\omega) = \frac{A_1}{3+j\omega} + \frac{A_2}{2+j\omega}\,,$$

with

$$A_1 = (3+j\omega)\,\hat{\mathbf{V}}_C(\omega)\big|_{j\omega=-3} = \frac{20}{2+j\omega}\bigg|_{j\omega=-3} = -20,$$

$$A_2 = (2+j\omega)\,\hat{\mathbf{V}}_C(\omega)\big|_{j\omega=-2} = \frac{20}{3+j\omega}\bigg|_{j\omega=-2} = 20.$$

Hence,

$$\hat{\mathbf{V}}_C(\omega) = \frac{-20}{3+j\omega} + \frac{20}{2+j\omega}\,,$$

and

$$\boxed{\upsilon_C(t) = 20(e^{-2t} - e^{-3t})\,u(t)\ \text{ V.}}$$

(c) $\upsilon_s(t) = 10 + 5\cos 4t$ V.

$$\hat{\mathbf{V}}_s(\omega) = 20\pi\,\delta(\omega) + 5\pi[\delta(\omega-4) + \delta(\omega+4)].$$

Hence,

$$\hat{\mathbf{V}}_C(\omega) = \frac{2}{3+j\omega}\,\hat{\mathbf{V}}_s(\omega)$$

$$= \frac{40\pi\,\delta(\omega)}{3+j\omega} + \frac{10\pi\,\delta(\omega-4)}{3+j\omega} + \frac{10\pi\,\delta(\omega+4)}{3+j\omega}$$

and

$$\upsilon_{\rm C}(t) = \boldsymbol{\mathfrak{F}}^{-1}[\hat{\mathbf{V}}_{\rm C}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\mathbf{V}}_{\rm C}(\omega)\, e^{j\omega t}\, d\omega$$

$$= 20\int_{-\infty}^{\infty} \frac{\delta(\omega)\, e^{j\omega t}}{3+j\omega}\, d\omega + 5\int_{-\infty}^{\infty} \frac{\delta(\omega-4)\, e^{j\omega t}}{3+j\omega}\, d\omega$$

$$+5\int_{-\infty}^{\infty} \frac{\delta(\omega+4)\, e^{j\omega t}}{3+j\omega}\, d\omega$$

$$= \frac{20}{3} + \left(\frac{5e^{j4t}}{3+j4} + \frac{5e^{-j4t}}{3-j4}\right)$$

$$= \boxed{\frac{20}{3} + 2\cos(4t - 36.9^\circ) \text{ V.}}$$

Exercise 6-1 Convert the following magnitude ratios to dB: (a) 20, (b) 0.03, (c) 6×10^6.

Solution:

(a) $20\log(20) = 20 \times 1.301 = \boxed{26.02 \text{ dB.}}$

(b) $20\log(0.03) = 20 \times (-1.523) = \boxed{-30.46 \text{ dB.}}$

(c) $20\log(6 \times 10^6) = 20\log 6 + 20\log 10^6 = 15.56 + 120 = \boxed{135.56 \text{ dB.}}$

Exercise 6-2 Convert the following dB values to magnitude ratios: (a) 36 dB, (b) -24 dB, (c) -0.5 dB.

Solution:

(a) $(10)^{36/20} = \boxed{63.1.}$

(b) $(10)^{-24/20} = \boxed{0.063.}$

(c) $(10)^{-0.5/20} = \boxed{0.94.}$

Exercise 6-3 Determine the order of $\hat{\mathbf{H}}(\omega) = \hat{\mathbf{V}}_{\text{out}}/\hat{\mathbf{V}}_{\text{s}}$ for the circuit in Fig. E6-3.

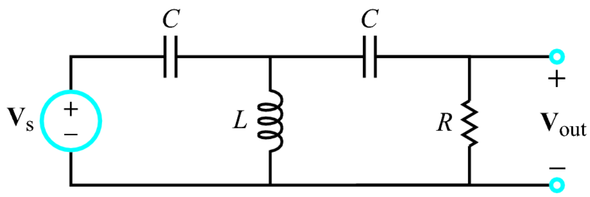


Figure E6-3

Solution: Circuit analysis leads to

$$\hat{\mathbf{H}}(\omega) = \frac{j\omega^3 RLC^2}{\omega^2 LC - (1-\omega^2 LC)(1+j\omega RC)} .$$

For ω very large, such that $\omega^2 LC \gg 1$ and $\omega RC \gg 1$,

$$\hat{\mathbf{H}}(\omega) \simeq 1, \qquad \omega \text{ very large.}$$

For ω very small, such that $\omega^2 LC \ll 1$ and $\omega RC \ll 1$,

$$\hat{\mathbf{H}}(\omega) \simeq j\omega^3 RLC^2.$$

Hence, filter is third order.

Exercise 6-4 Choose values for R_s and R_f in the circuit of Fig. 6-16(b) so that the gain magnitude is 10 and the corner frequency is 10^3 rad/s, given that $C_f = 1\ \mu$F.

Solution: According to Eq. (6.51),

$$G_{LP} = -\frac{R_f}{R_s} = -10,$$

$$\omega_{LP} = \frac{1}{R_f C_f} = 10^3 \text{ rad/s}.$$

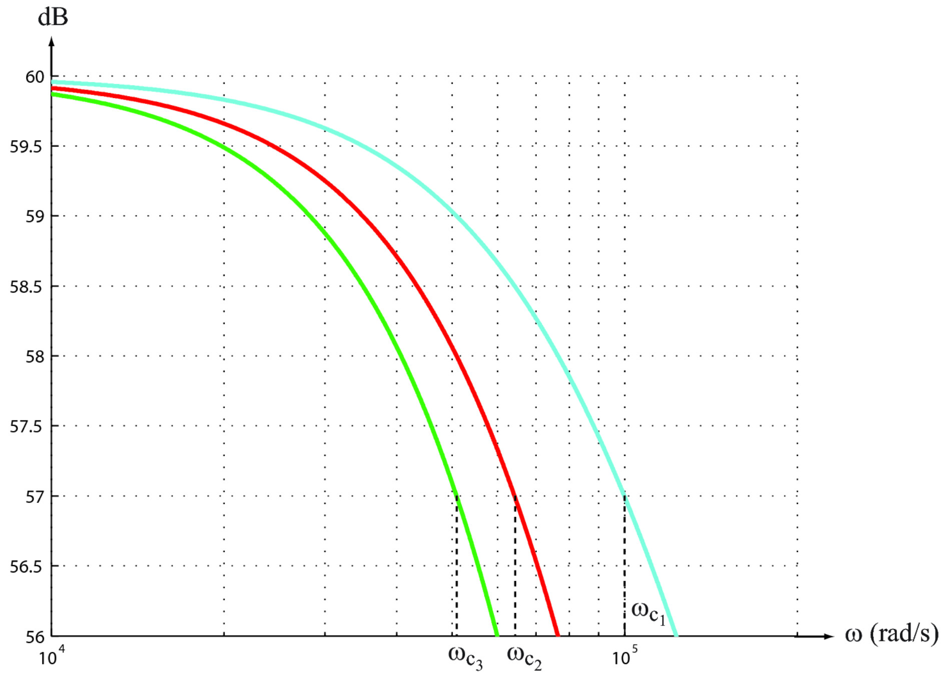
With $C_f = 1\ \mu$F,

$$\boxed{R_f = 1\ \text{k}\Omega,} \qquad \text{and} \qquad \boxed{R_s = 100\ \Omega.}$$

Exercise 6-5 What are the values of the corner frequencies associated with M_1, M_2, and M_3 of Example 6-4?

Solution: By plotting the expressions for M_1, M_2, and M_3 and determining the angular frequencies at which each is $1/\sqrt{2}$ of its peak value, we can show that

$$\omega_{c_1} = 10^5 \text{ rad/s}, \quad \omega_{c_2} = 0.64\omega_{c_1}, \quad \text{and } \omega_{c_3} = 0.51\omega_{c_1}.$$



Exercise 6-6 Determine the output from a brick-wall lowpass filter with a cutoff frequency of 0.2 Hz, given that the input is the square wave given by Eq. (6.57).

Solution: This exercise is similar to Example 6-5.

The square wave given by Eq. (6.57) has the Fourier series given by Eq. (6.58):

$$x(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \cdots ,$$

which has components at frequencies $\frac{1}{2\pi} = 0.16$ Hz, $\frac{3}{2\pi} = 0.48$ Hz, $\frac{5}{2\pi} = 0.80$ Hz, etc.

The brick-wall lowpass filter with a cutoff frequency of 0.2 Hz will allow only the first component to pass through. Hence the output is simply

$$\boxed{y(t) = \sin(t).}$$

Exercise 6-7 Determine the output from a brick-wall bandpass filter with $f_{c_1} = 0.2$ Hz and $f_{c_2} = 1$ Hz, given that the input is the square wave given by Eq. (6.57).

Solution: This exercise is similar to Example 6-5.

The square wave given by Eq. (6.57) has the Fourier series given by Eq. (6.58):

$$x(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) + \cdots ,$$

which has components at $\frac{1}{2\pi} = 0.16$ Hz, $\frac{3}{2\pi} = 0.48$ Hz, $\frac{5}{2\pi} = 0.80$ Hz, $\frac{7}{2\pi} = 1.12$ Hz, etc.

The brick-wall bandpass filter with cutoff frequencies 0.2 Hz and 1 Hz will allow only the second and third components to pass through, since $0.16 < 0.2$ Hz, $0.2 < 0.48$, $0.80 < 1$ Hz, and $1.12 > 1$ Hz. Hence the output is simply

$$\boxed{y(t) = \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t).}$$

Exercise 6-8 An LTI system has zeros at $\pm j3$. What sinusoidal signals will it eliminate?

Solution: An LTI system with zeros at $\pm j3$ has a transfer function of the form

$$\mathbf{H}(\mathbf{s}) = (\mathbf{s} - j3)(\mathbf{s} + j3)\,\mathbf{H}_0(\mathbf{s})$$

for some rational function $\mathbf{H}_0(\mathbf{s})$.

The frequency response of the system has the form

$$\hat{\mathbf{H}}(\omega) = (j\omega - j3)(j\omega + j3)\,\hat{\mathbf{H}}_0(\omega)$$

for some rational function $\hat{\mathbf{H}}_0(\omega)$.

The response to a general sinusoid of the form $x(t) = A\cos(3t + \theta)$ is then

$$y(t) = A|\hat{\mathbf{H}}(3)|\cos(3t + \theta + \angle[\hat{\mathbf{H}}(3)]) = 0$$

because $\hat{\mathbf{H}}(3) = 0$. Hence, the system will eliminate $x(t) = A\cos(3t + \theta)$ for any values of A or θ.

Exercise 6-9 An LTI system has poles at $-0.1 \pm j4$. What sinusoidal signals will it emphasize?

Solution: An LTI system with poles at $(-0.1 \pm j4)$ has a transfer function of the form

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{H}_0(\mathbf{s})}{(\mathbf{s}+0.1+j4)(\mathbf{s}+0.1-j4)} \,,$$

where $\mathbf{H}_0(\mathbf{s})$ is some rational function. The frequency response of the system is

$$\hat{\mathbf{H}}(\omega) = \mathbf{H}(\mathbf{s})|_{\mathbf{s}=j\omega} = \frac{\hat{\mathbf{H}}_0(\omega)}{[0.1+j(\omega+4)][0.1+j(\omega-4)]} \,.$$

Hence, $\hat{\mathbf{H}}(\omega)$ wil emphasize any sinusoid $x(t) = A\cos(\omega t + \theta)$ at $\omega = 4$ rad/s.

Exercise 6-10 Design (specify the transfer function of) a notch filter to reject a 50-Hz sinusoid. The filter's impulse response must decay to 0.005 within 6 seconds.

Solution: This exercise is similar to Example 6-7. We have $\omega_0 = 2\pi \times 50 = 100\pi$ rad/s, but we need to find α.

The amplitude of the cosine term in $h_{\text{notch}}(t)$ is $Ae^{-\alpha t}$. At $t = 6$ s, we require

$$Ae^{-6\alpha} = \left[4\alpha^2 + \frac{\alpha^4}{\omega_0^2}\right]^{1/2} e^{-6\alpha} < 0.005.$$

Trial and error leads to $\alpha = 1$ s^{-1} as the solution. Inserting $\alpha = 1$ s^{-1} and $\omega_0 = 100\pi$ rad/s gives

$$\mathbf{H}(\mathbf{s}) = 1 - \frac{2\alpha\mathbf{s}}{(\mathbf{s}+\alpha)^2 + \omega_0^2} = \boxed{1 - \frac{2\mathbf{s}+1}{\mathbf{s}^2 + 2\mathbf{s} + 98697}}\,.$$

Exercise 6-11 Design a comb filter to eliminate periodic interference with period = 1 ms. Assume that harmonics above 2 kHz are negligible. Use $\alpha = 100$ s^{-1}.

Solution: We need a comb filter that can eliminate 1-kHz and 2-kHz sinusoids. Hence, $n = 2$, which matches Eq. (6.80). Upon setting $\omega_0 = 2000\pi$ rad/s and $\alpha = 100$ s^{-1}, and replacing $j\omega$ with $\mathbf{s}$, Eq. (6.80) becomes

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{s}^2 + (2000\pi)^2}{\mathbf{s}^2 + 200\mathbf{s} + 10^4 + (2000\pi)^2} \times \frac{\mathbf{s}^2 + (4000\pi)^2}{\mathbf{s}^2 + 200\mathbf{s} + 10^4 + (4000\pi)^2} \ .$$

Exercise 6-12 Where should the poles of a second-order Butterworth lowpass filter be located, if its cutoff frequency is 3 rad/s?

Solution: Since the order 2 is even, distribute $2(2) = 4$ equally spaced poles around the circle of radius 3 rad/s, symmetrically arranged with respect to both axes. These poles are at $\{3e^{\pm j45^\circ}, 3e^{\pm j135^\circ}\}$. Discarding the poles in the right half-plane leaves

$$\boxed{\text{poles at } \{3e^{\pm j135^\circ}\}.}$$

Exercise 6-13 Where should the poles of a third-order Butterworth lowpass filter be located, if its cutoff frequency is 5 rad/s?

Solution: Since the order 3 is odd, distribute $2(3) = 6$ equally spaced pole around the circle of radius 5 rad/s, starting with $5e^{j0}$. These poles are at

$$\{5e^{j0}, 5e^{\pm j60^\circ}, 5e^{\pm j120^\circ}, 5e^{j180^\circ}\}.$$

Discarding the poles in the right half-plane leaves

$$\boxed{\text{poles at } \{-5, 5e^{\pm j120^\circ}\}.}$$

Exercise 6-14 Obtain the transfer function of a resonator filter designed to enhance 5-Hz sinusoids. Use $\alpha = 2$.

Solution: The transfer function of a resonator filter is

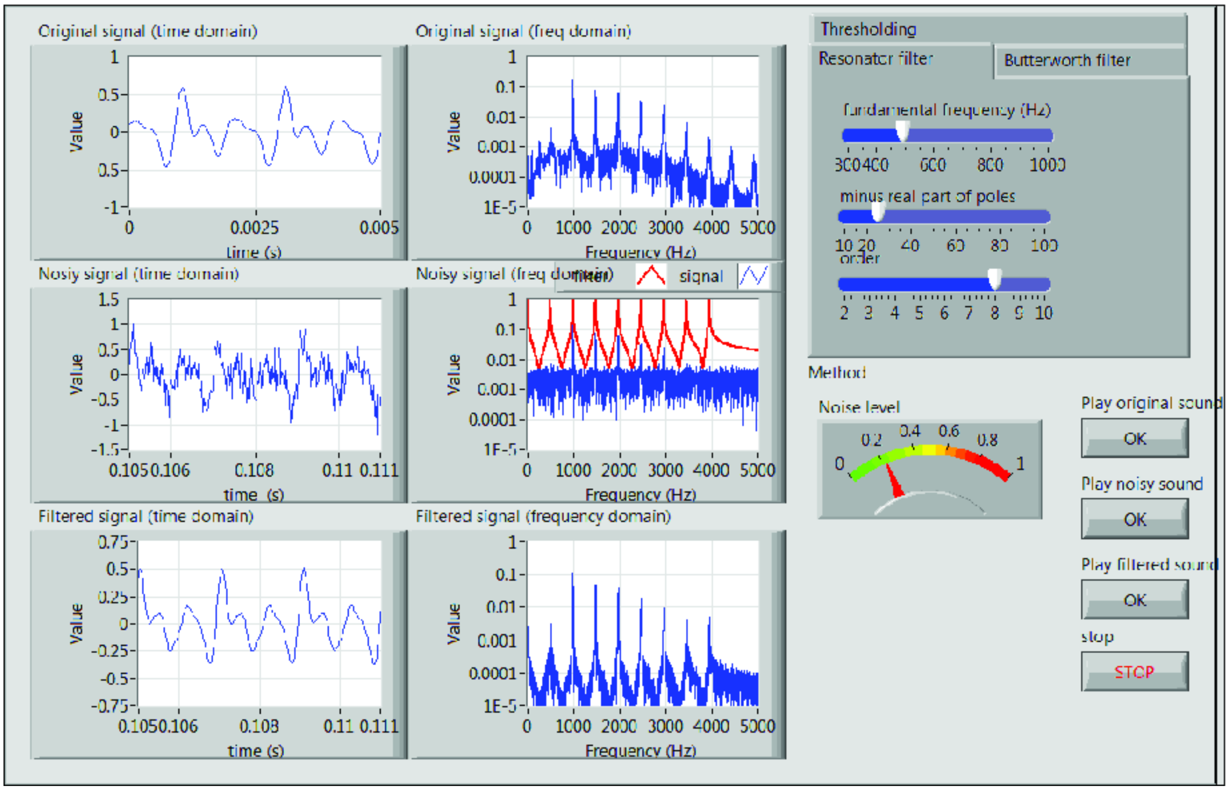
$$\mathbf{H}_{\text{resonator}}(\mathbf{s}) = 1 - \mathbf{H}_{\text{notch}}(\mathbf{s}) = \frac{2\alpha\mathbf{s} + \alpha^2}{(\mathbf{s}+\alpha)^2 + \omega_0^2}\,.$$

Inserting $\omega_0 = 10\pi$ rad/s and $\alpha = 2\ \text{s}^{-1}$ gives

$$\mathbf{H}_{\text{resonator}}(\mathbf{s}) = \boxed{\frac{4\mathbf{s}+4}{\mathbf{s}^2 + 4\mathbf{s} + 991}\,.}$$

Exercise 6-15 Use LabVIEW Module 6.3 to denoise the noisy trumpet signal using a resonator filter, following Example 6-13. Use a noise level of 0.2.

Solution:



Exercise 6-16 Given 20 signals, each of (two-sided) bandwidth $B_b = 10$ kHz, how much total bandwidth would be needed to combine them using FDM with SSB modulation and no guard bands between adjacent signals?

Solution: With SSB modulation, only half of the two-sided bandwidth is used, which in the present case translates into 5 kHz per signal. For 20 FDM-combined signals with no guard bands between them, the total bandwidth is

$$20 \times 5 \text{ kHz} = \boxed{100 \text{ kHz.}}$$

Exercise 6-17 Figure E6-17 depicts the frequency bands allocated by the U.S. Federal Communications Commission (FCC) to four AM radio stations. Each band is 8 kHz in extent. Suppose radio station WJR (with carrier frequency of 760 kHz) were to accidentally transmit a 7-kHz tone, what impact might that have on other stations? [Even though the four stations are separated by long distances, let us assume they are close to one another.]

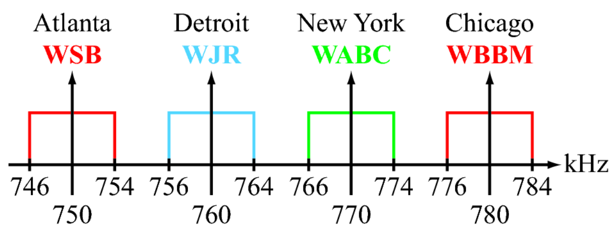


Figure E6-17

Solution: After modulation by 760 kHz, the tone has frequencies at:

$$760-7=753 \text{ kHz}, \qquad \text{and} \qquad 760+7=767 \text{ kHz}.$$

Listeners tuning in to stations at 750 kHz and 770 kHz will hear a tone at 3 kHz, because

$$753-750=3 \text{ kHz}, \qquad \text{and} \qquad 767-770=-3 \text{ kHz}.$$

Remember that frequency bands centered at -750 and -770 kHz are also modulated to baseband, and $-767-(-770)=3$ kHz.

Exercise 6-18 What is the Nyquist sampling rate for a signal bandlimited to 5 kHz?

Solution: The Nyquist rate is double the maximum frequency.

$$2(5 \text{ kHz}) = \boxed{10000 \text{ samples/s.}}$$

Exercise 6-19 A 500 Hz sinusoid is sampled at 900 Hz. No anti-alias filter is used. What is the frequency of the reconstructed sinusoid?

Solution:

The spectrum of the sampled signal has components at frequencies

$$\{\pm 500, \pm 900 \pm 500, \pm 1800 \pm 500, \dots\} = \{\pm 400, \pm 500, \pm 1300, \pm 1400, \dots\} \text{ Hz.}$$

The reconstruction filter is a lowpass filter with cutoff at $\frac{1}{2}(900) = 450$ Hz.

This leaves components at ± 400 Hz. The frequency of the reconstructed sinusoid is

$$\boxed{400 \text{ Hz.}}$$

Exercise 7-1 Determine the duration of $\{3, \underline{1}, 4, 6\}$.

Solution: In discrete time, the duration of a signal that is zero outside the interval $a \leq n \leq b$ is $b - a + 1$.

Here, $a = -1$, $b = 2$, so the duration is $b - a + 1 = \boxed{4.}$

Exercise 7-2 If the mean value of $x[n]$ is 3, what transformation results in a zero-mean signal?

Solution: The mean of the sum of two signals is the sum of the means.

$$\boxed{y[n] = x[n] - 3.}$$

Exercise 7-3 Determine the fundamental period and fundamental angular frequency of

$$3\cos(0.56\pi n + 1).$$

Solution: According to Eq. (7.21),

$$N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{0.56\pi} = \frac{25k}{7}.$$

Hence, select $k = 7$ and $\boxed{N_0 = 25 \text{ samples.}}$

Also,

$$\Omega_0 = \frac{2\pi}{N_0} = \boxed{\frac{2\pi}{25} \text{ rad/sample.}}$$

Exercise 7-4 Compute the fundamental angular frequency of $2\cos(5.1\pi n+1)$.

Solution: According to Eq. (7.21),

$$N_0 = \frac{2\pi k}{\Omega} = \frac{2\pi k}{5.1\pi} = \frac{20k}{51}\ .$$

Hence, select $k = 51$ and $\boxed{N_0 = 20 \text{ samples.}}$

$$\Omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{20} = \boxed{0.1\pi \text{ rad/sample.}}$$

Exercise 7-5 Transform the following equation into the form of an ARMA difference equation:

$$y[n+2]+2y[n]=3x[n+1]+4x[n-1].$$

Solution: The output $y[n]$ must be a linear combination of $y[n-i]$, $x[n]$ and $x[n-i]$ for $i>0$.

Replacing n with $n-2$ gives $\boxed{y[n]+2y[n-2]=3x[n-1]+4x[n-3].}$

Exercise 7-6 Is the system with impulse response

$$h[n] = \frac{1}{(n+1)^2}\, u[n]$$

BIBO stable?

Solution: An LTI system is BIBO stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]|$ is finite.

Here,

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}\ .$$

This is finite, so the system $\boxed{\text{is BIBO stable.}}$

Exercise 7-7 Is the system with $h[n] = \left(\frac{1}{2}\right)^n u[n]$ BIBO stable?

Solution: An LTI system with impulse response $h[n] = \mathbf{C}\mathbf{p}^n\, u[n]$ is BIBO stable if and only if $|\mathbf{p}| < 1$, since

$$\sum_{n=-\infty}^{\infty} |h[n]| = |\mathbf{C}| \sum_{n=0}^{\infty} |\mathbf{p}|^n = \frac{|C|}{1-|\mathbf{p}|}$$

if and only if $|\mathbf{p}| < 1$.

Here, $\mathbf{p} = 0.5$ and $|\mathbf{p}| = |0.5| < 1$, so the system $\boxed{\text{is BIBO stable.}}$

Exercise 7-8 A system has an impulse response $h[n] = 0$ for $n < 0$ but $h[0] \neq 0$. Is the system causal?

Solution:

$$y[n] = \sum_{i=0}^{\infty} h[i]\, x[n-i] = h[0]\, x[n] + h[1]\, x[n-1] + \cdots .$$

$h[0] \neq 0$ means that $y[n]$ depends on $x[n]$ as well as on past values of $x[n]$.
The output of a causal system at time n can depend on the input at the same time n and at previous times.

Hence, the system is $\boxed{\text{causal.}}$

Exercise 7-9 Compute $y[n] = \{1, \underline{2}\} * \{\underline{0}, 0, 3, 4\}$.

Solution: First, note that $\{\underline{1}, 2\} * \{\underline{3}, 4\} = \{(1)(3), (1)(4) + (2)(3), (2)(4)\} = \{\underline{3}, 10, 8\}$.

Using property #5 with $a = -1$ and $b = 2$ gives $y[n] = \boxed{\{\underline{0}, 3, 10, 8\}.}$

Exercise 7-10 Compute the $\mathbf{z}$-transform of $\{\underline{1}, 2\}$. Put the answer in the form of a rational function.

Solution:

$$\mathbb{Z}[\{\underline{1},2\}] = 1\mathbf{z}^{-0} + 2\mathbf{z}^{-1} = \boxed{\frac{\mathbf{z}+2}{\mathbf{z}}}.$$

Exercise 7-11 Compute the $\mathbf{z}$-transform of $\{\underline{1},1\}+(-1)^n\,u[n]$. Put the answer in the form of a rational function.

Solution:

$$\mathbb{Z}[\{\underline{1},1\}] = 1\mathbf{z}^{-0}+1\mathbf{z}^{-1},$$

$$\mathbb{Z}[(-1)^n\,u[n]] = \frac{\mathbf{z}}{\mathbf{z}-(-1)} = \frac{\mathbf{z}}{\mathbf{z}+1},$$

$$\mathbb{Z}[\{\underline{1},1\}+(-1)^n\,u[n]] = 1+\mathbf{z}^{-1}+\frac{\mathbf{z}}{\mathbf{z}+1} = \frac{\mathbf{z}+1}{\mathbf{z}}+\frac{\mathbf{z}}{\mathbf{z}+1} = \boxed{\frac{2\mathbf{z}^2+2\mathbf{z}+1}{\mathbf{z}^2+\mathbf{z}}}\,.$$

Exercise 7-12 Compute $\mathbb{Z}[n\, u[n]]$, given that $\mathbb{Z}[u[n]] = \frac{\mathbf{z}}{\mathbf{z}-1}$.

Solution: Using the **z**-derivative property,

$$\mathbb{Z}[n\, u[n]] = -\mathbf{z}\,\frac{d}{d\mathbf{z}}\left[\frac{\mathbf{z}}{\mathbf{z}-1}\right] = \boxed{\frac{\mathbf{z}}{(\mathbf{z}-1)^2}\,.}$$

Exercise 7-13 Compute $\mathbb{Z}[na^n\, u[n]]$, given that

$$\mathbb{Z}[n\, u[n]] = \frac{\mathbf{z}}{(\mathbf{z}-1)^2}\ .$$

Solution: Using the $\mathbf{z}$-scaling property,

$$\mathbb{Z}[na^n\, u[n]] = \frac{\mathbf{z}/a}{((\mathbf{z}/a)-1)^2}\frac{a^2}{a^2} = \boxed{\frac{a\mathbf{z}}{(\mathbf{z}-a)^2}}\ .$$

Exercise 7-14 Compute the inverse **z**-transform of $(\mathbf{z}+3)/(\mathbf{z}+1)$.

Solution:

$$\frac{\mathbf{z}+3}{\mathbf{z}+1} = 1 + \frac{2}{\mathbf{z}+1},$$

$$\mathbb{Z}^{-1}\left[\frac{\mathbf{z}+3}{\mathbf{z}+1}\right] = \mathbb{Z}^{-1}[1] + \mathbb{Z}^{-1}\left[\frac{2}{\mathbf{z}+1}\right] = \boxed{\delta[n] + 2(-1)^{n-1}\, u[n-1].}$$

Exercise 7-15 Compute the inverse **z**-transform of $1/[(\mathbf{z}+1)(\mathbf{z}+2)]$.

Solution: The partial fraction expansion is

$$\frac{1}{(\mathbf{z}+1)(\mathbf{z}+2)} = \frac{1}{\mathbf{z}+1} - \frac{1}{\mathbf{z}+2}\,.$$

The inverse **z**-transform is then $\boxed{(-1)^{n-1}\, u[n-1] - (-2)^{n-1}\, u[n-1].}$

Exercise 7-16 Use **z**-transforms to compute the zero-input response of the system

$$y[n]-2y[n-1]=3x[n]+4x[n-1]$$

with initial condition $y[-1]=\frac{1}{2}$.

Solution: Zero-input response means $x[n]=x[n-1]=0$. Hence, the system equation reduces to

$$y[n]-2y[n-1]=0.$$

Transferring to the **z**-domain:

$$y[n] \quad \longleftrightarrow \quad \mathbf{Y}(\mathbf{z})$$

$$y[n-1] \quad \longleftrightarrow \quad \frac{1}{\mathbf{z}}\,\mathbf{Y}(\mathbf{z})+y[-1]=\frac{1}{\mathbf{z}}\,\mathbf{Y}(\mathbf{z})+\frac{1}{2}\,.$$

Hence,

$$\mathbf{Y}(\mathbf{z})-2\left[\frac{1}{\mathbf{z}}\,\mathbf{Y}(\mathbf{z})+\frac{1}{2}\right]=0,$$

$$\mathbf{Y}(\mathbf{z})\left[1-\frac{2}{\mathbf{z}}\right]=1,$$

$$\mathbf{Y}(\mathbf{z})=\frac{1}{1-\frac{2}{\mathbf{z}}}=\frac{\mathbf{z}}{\mathbf{z}-2}\,.$$

From Table 7-5,

$$\boxed{y[n]=2^n\,u[n].}$$

Exercise 7-17 A system is described by

$$y[n] - \frac{3}{4}\, y[n-1] + \frac{1}{8}\, y[n-2] = x[n] + 2x[n-1].$$

Compute its transfer function.

Solution: Taking the **z**-transform gives

$$\mathbf{Y}(\mathbf{z}) \left[1 - \frac{3}{4}\mathbf{z}^{-1} + \frac{1}{8}\mathbf{z}^{-2}\right] = \mathbf{X}(\mathbf{z})\; [1 + 2\mathbf{z}^{-1}].$$

Hence,

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})} = \frac{1 + 2\mathbf{z}^{-1}}{1 - \frac{3}{4}\mathbf{z}^{-1} + \frac{1}{8}\mathbf{z}^{-2}}\, \frac{\mathbf{z}^2}{\mathbf{z}^2} = \boxed{\frac{\mathbf{z}^2 + 2\mathbf{z}}{\mathbf{z}^2 - \frac{3}{4}\,\mathbf{z} + \frac{1}{8}}}\,.$$

Exercise 7-18 A system is described by

$$y[n] - \frac{3}{4}\,y[n-1] + \frac{1}{8}\,y[n-2] = x[n] + 2x[n-1].$$

Determine its poles and zeroz and whether or not it is BIBO stable.

Solution: Taking the $\mathbf{z}$-transform gives

$$\mathbf{Y}(\mathbf{z})\left[1 - \frac{3}{4}\mathbf{z}^{-1} + \frac{1}{8}\mathbf{z}^{-2}\right] = \mathbf{X}(\mathbf{z})[1 + 2\mathbf{z}^{-1}].$$

Hence,

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})} = \frac{1 + 2\mathbf{z}^{-1}}{1 - \frac{3}{4}\mathbf{z}^{-1} + \frac{1}{8}\mathbf{z}^{-2}}\,\frac{\mathbf{z}^2}{\mathbf{z}^2} = \frac{\mathbf{z}^2 + 2\mathbf{z}}{\mathbf{z}^2 - \frac{3}{4}\,\mathbf{z} + \frac{1}{8}} = \frac{(\mathbf{z}-0)(\mathbf{z}+2)}{(\mathbf{z}-\frac{1}{2})(\mathbf{z}-\frac{1}{4})}\,.$$

The system has $\boxed{\text{zeros } \{0, -2\}}$ and $\boxed{\text{poles } \{\frac{1}{2}, \frac{1}{4}\}.}$ Note that $|\frac{1}{2}| < 1$ and $|\frac{1}{4}| < 1$.

Since both poles are inside the unit circle, the system $\boxed{\text{is BIBO stable.}}$

Exercise 7-19 Compute the response of the system $y[n] = x[n] - x[n-2]$ to input $x[n] = \cos(\pi n/4)$.

Solution: Take the **z**-transform:

$$\mathbf{Y(z)} = \mathbf{X(z)} - \mathbf{z}^{-2}\,\mathbf{X(z)} = [1 - \mathbf{z}^{-2}]\,\mathbf{X(z)}.$$

The transfer function is

$$\mathbf{H(z)} = \frac{\mathbf{Y(z)}}{\mathbf{X(z)}} = 1 - \mathbf{z}^{-2}\,.$$

Substituting $\mathbf{z} = e^{j\Omega}$ gives the frequency response function

$$\mathbf{H}(e^{j\Omega}) = 1 - e^{-j2\Omega}\,.$$

Substituting $\Omega = \frac{\pi}{4}$ gives

$$\mathbf{H}(e^{j\pi/4}) = 1 - e^{-j\pi/2} = 1 - (-j) = 1 + j = \sqrt{2}\,e^{j\pi/4}\,.$$

The response of the system to $x[n]$ is $\boxed{1.414\cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right).}$

Exercise 7-20 An LTI system has $\mathbf{H}(e^{j\Omega}) = j\tan(\Omega)$. Compute the difference equation.

Solution: We need to write $\mathbf{H}(e^{j\Omega})$ as a function of $e^{j\Omega}$, not just of Ω.

Using the definitions

$$2\cos(\Omega) = e^{j\Omega} + e^{-j\Omega}$$

and

$$2j\sin(\Omega) = e^{j\Omega} - e^{-j\Omega},$$

gives

$$\mathbf{H}(e^{j\Omega}) = j\tan(\Omega) = \frac{2j\sin(\Omega)}{2\cos(\Omega)} = \frac{e^{j\Omega} - e^{-j\Omega}}{e^{j\Omega} + e^{-j\Omega}}\ .$$

Substituting $e^{j\Omega} = \mathbf{z}$ gives the transfer function

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{z} - \mathbf{z}^{-1}}{\mathbf{z} + \mathbf{z}^{-1}} \frac{\mathbf{z}^{-1}}{\mathbf{z}^{-1}} = \frac{1 - \mathbf{z}^{-2}}{1 + \mathbf{z}^{-2}} = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})}\ .$$

Cross-multiplying gives

$$\mathbf{Y}(\mathbf{z})\ [1 + \mathbf{z}^{-2}] = \mathbf{X}(\mathbf{z})\ [1 - \mathbf{z}^{-2}].$$

An inverse **z**-transform gives $\boxed{y[n] + y[n-2] = x[n] - x[n-2].}$

Exercise 7-21 Compute the DTFS of $4\cos(0.15\pi n+1)$.

Solution: According to Eq. (7.136a), the DTFS representation is given by

$$x[n] = \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{jk\Omega_0 n}$$
$$= \sum_{k=0}^{N_0-1} \mathbf{x}_k e^{j2\pi nk/N_0}, \tag{0.1}$$

where we used the relationship $\Omega_0 = 2\pi/N_0$. Our goal is to find the values of $\mathbf{x}_k$, which we can we do by applying Eq. (7.136b) or by comparing Eq. (1) with the given sinusoid after expressing the latter in terms of complex exponentials. The second approach entails writing $x[n]$ as

$$\begin{aligned} x[n] &= 4\cos(0.15\pi n+1) \\ &= 4\cos\left(2\pi\left(\frac{3}{40}\right)n+1\right) \\ &= 2\cos\left[e^{j\left(2\pi\left(\frac{3}{40}\right)n+1\right)} + e^{-j\left(2\pi\left(\frac{3}{40}\right)n+1\right)}\right] \\ &= 2e^{j1}e^{j2\pi\left(\frac{3}{40}\right)n} + 2e^{-j1}e^{-j2\pi\left(\frac{3}{40}\right)n}e^{j2\pi} \\ &= 2e^{j1}e^{j2\pi\left(\frac{3}{40}\right)n} + 2e^{-j1}e^{j2\pi\left(1-\frac{3}{40}\right)n} \\ &= 2e^{j1}e^{j2\pi\left(\frac{3}{40}\right)n} + 2e^{-j1}e^{j2\pi\left(\frac{37}{40}\right)n}. \end{aligned} \tag{0.2}$$

We surmise from the expression for $x[n]$ that $N_0 = 40$ samples. Comparison of the two terms in Eq. (2) with the summation in Eq. (1) leads to the conclusion that the first term corresponds to $k = 3$ and the second term corresponds to $k = 37$. Hence,

$$\boxed{\mathbf{x}_3 = 2e^{j1}, \quad \mathbf{x}_{37} = 2e^{-j1},}$$

and all other terms for $k = 0$ to 39 are zero.

Exercise 7-22 Confirm Parseval's rule for the above exercise.

Solution: Parseval's theorem states that the average power of a periodic signal is the same whether it is computed in the time domain or in the frequency (DTFS) domain.

Time domain: The average power of the *periodic* sinusoid is $\boxed{\frac{4^2}{2} = 8.}$

Frequency domain: The average powers of the two *periodic* complex exponentials are $\boxed{|2e^{j1}|^2 + |2e^{-j1}|^2 = 8.}$

Hence, the average powers are identical.

Exercise 7-23 Compute the DTFT of $4\cos(0.15\pi n+1)$.

Solution: According to entry #6 in Table 7-8, the DTFT of $A\cos(\Omega_0 n+\theta)$ is

$$A\pi e^{j\theta}\,\delta((\Omega-\Omega_0))+A\pi e^{-j\theta}\,\delta((\Omega+\Omega_0)),$$

where

$$\delta((\Omega-\Omega_0))=\sum_{k=-\infty}^{\infty}\delta(\Omega+2\pi k-\Omega_0)$$

is a chain of impulses in Ω.

Note that the DTFT of any signal is *always* periodic in Ω with period 2π.

Here, $A=4$, $\Omega_0=0.15\pi$, $\theta=1$, so the DTFT is

$$\boxed{4\pi e^{j1}\delta((\omega-0.15))+4\pi e^{-j1}\delta((\omega+0.15)).}$$

Exercise 7-24 Compute the inverse DTFT of

$$4\cos(2\Omega)+6\cos(\Omega)+j8\sin(2\Omega)+j2\sin(\Omega).$$

Solution: We need to write this function $\mathbf{X}(e^{j\Omega})$ in terms of $e^{j\Omega}$, not just of Ω.

Using the definitions

$$2\cos(\Omega)=e^{j\Omega}+e^{-j\Omega}$$

and

$$2j\sin(\Omega)=e^{j\Omega}-e^{-j\Omega}$$

gives

$$\mathbf{X}(e^{j\Omega})=[2e^{j2\Omega}+2e^{-j2\Omega}]+[3e^{j\Omega}+3e^{-j\Omega}]+[4e^{j2\Omega}-4e^{-j2\Omega}]+[e^{j\Omega}-e^{-j\Omega}].$$

Summing terms,

$$\mathbf{X}(e^{j\Omega})=6e^{j2\Omega}+4e^{j\Omega}+2e^{-j\Omega}-2e^{-j2\Omega}.$$

The DTFT is

$$\mathbf{X}(e^{j\Omega})=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\Omega n},$$

so we can read off $\boxed{\{6,4,\underline{0},2,-2\}.}$

Exercise 7-25 Compute the 4-point DFT of $\{4,3,2,1\}$.

Solution: The N_0-point DFT is

$$\mathbf{X}_k = \sum_{n=0}^{N_0-1} x[n]\, e^{-j2\pi nk/N_0} \qquad \text{for } k = 0,1,\ldots,N_0-1.$$

Here, $N_0 = 4$, and $e^{-j2\pi nk/4} = (-j)^{nk}$ for $k = 0,1,2,3$.

Hence,

$$\begin{aligned}
\mathbf{X}_0 &= x[0](1)+x[1](1)+x[2](1)+x[3](1) = 4+3+2+1 = 10,\\
\mathbf{X}_1 &= x[0](1)+x[1](-j)+x[2](-1)+x[3](j) = 4-j3-2+j1 = 2-j2,\\
\mathbf{X}_2 &= x[0](1)+x[1](-1)+x[2](1)+x[3](-1) = 4-3+2-1 = 2,\\
\mathbf{X}_3 &= x[0](1)+x[1](j)+x[2](-1)+x[3](-j) = 4+j3-2-j1 = 2+j2.
\end{aligned}$$

Since $x[n]$ is real-valued, we could also have used $\mathbf{X}_3 = \mathbf{X}_1^*$.

The DFT is $\boxed{\{10,(2-j2),2,(2+j2)\}.}$ **Check:** `fft([4 3 2 1])` gives the same answer.

Exercise 7-26 How many MADs are needed to compute a 4096-point DFT using the FFT?

Solution: $\frac{4096}{2}\log_2(4096) = 24576.$

Exercise 7-27 Using the decimation-in-frequency FFT, which values of the 8-point DFT of a signal of the form $\{a, b, c, d, e, f, g, h\}$ do not have a factor of $\sqrt{2}$ in them?

Solution: In the decimation-in-frequency FFT, the twiddle multiplications only affect the odd-valued indices. So $\{X_0, X_2, X_4, X_6\}$ do not have a factor of $\sqrt{2}$ in them.

Exercise 7-28 Using the decimation-in-time FFT, show that only two values of the 8-point DFT of a signal of the form $\{a,b,a,b,a,b,a,b\}$ are nonzero.

Solution: In the decimation-in-time FFT, 4-point DFTs of $\{a,a,a,a\}$ and $\{b,b,b,b\}$ are computed. These are both zero except for the dc ($k = 0$) values. So the 8-point DFT has only two nonzero values $X_0 = 4a+4b$ and $X_4 = 4a-4b$.

Exercise 8-1 Obtain the transfer function of a BIBO-stable, discrete-time lowpass filter consisting of a single pole and a single zero, given that the zero is on the unit circle, the pole is at a location within 0.001 from the unit circle, and the dc gain at $\Omega = 0$ is 1.

Solution: A zero at $e^{j\Omega_0}$ produces a dip in the magnitude $|\mathbf{H}(e^{j\Omega})|$ at $\Omega = \Omega_0$. A pole at $ae^{j\Omega_0}$ produces a peak in the magnitude $|\mathbf{H}(e^{j\Omega})|$ at $\Omega = \Omega_0$, where $a \approx 1$, but $a < 1$ is needed to make the system BIBO stable.

The lowpass filter should reject the highest discrete-time fundamental frequency $\Omega = \pi$, and pass the lowest discrete-time fundamental frequency $\Omega = 0$ (dc) with $\mathbf{H}(e^{j0}) = 1$. So it should have a zero at $e^{j\pi} = -1$ and a pole at $ae^{j0} = a$, for $a \approx 1$ and $a < 1$.

Using $a = 0.999$ gives

$$\mathbf{H}(\mathbf{z}) = \mathbf{C}\,\frac{\mathbf{z}+1}{\mathbf{z}-0.999}\,.$$

Also,

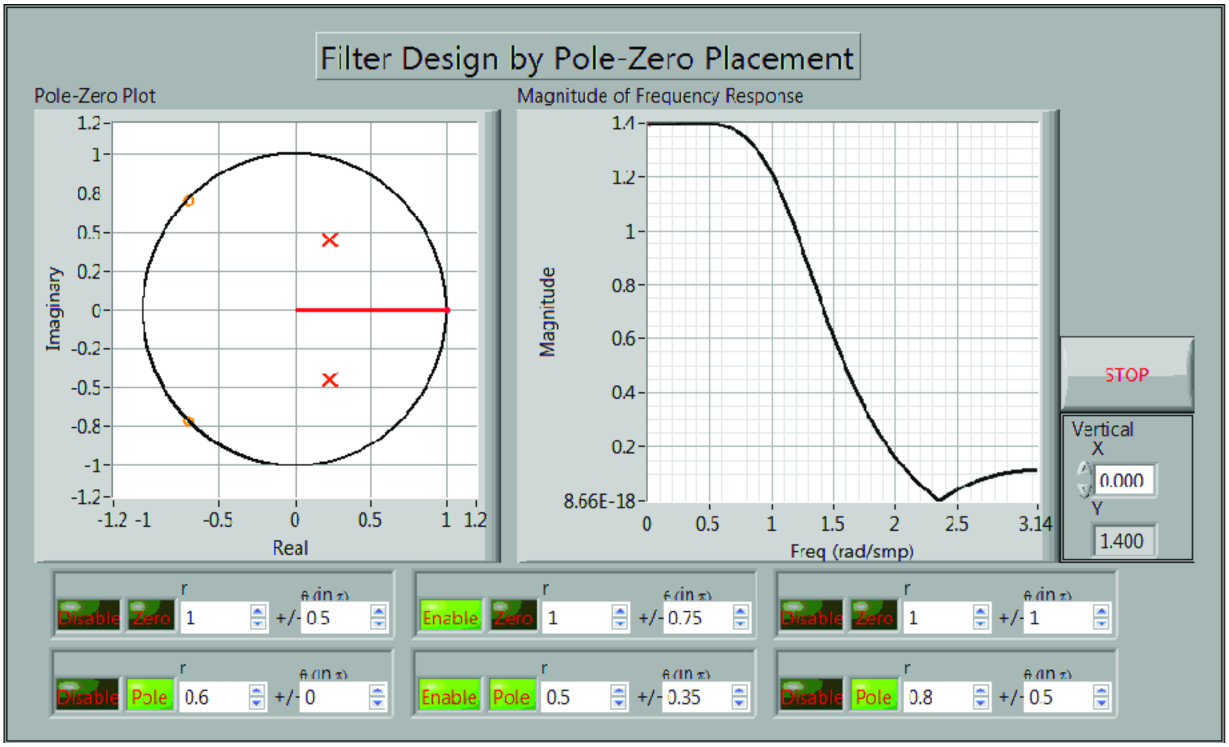
$$1 = \mathbf{H}(e^{j0}) = \mathbf{H}(1) = \mathbf{C}\,\frac{1+1}{1-0.999}\,.$$

Solving for $\mathbf{C}$ gives $\mathbf{C} = 0.0005$. Hence,

$$\boxed{\mathbf{H}(\mathbf{z}) = 0.0005\,\frac{\mathbf{z}+1}{\mathbf{z}-0.999}}\,.$$

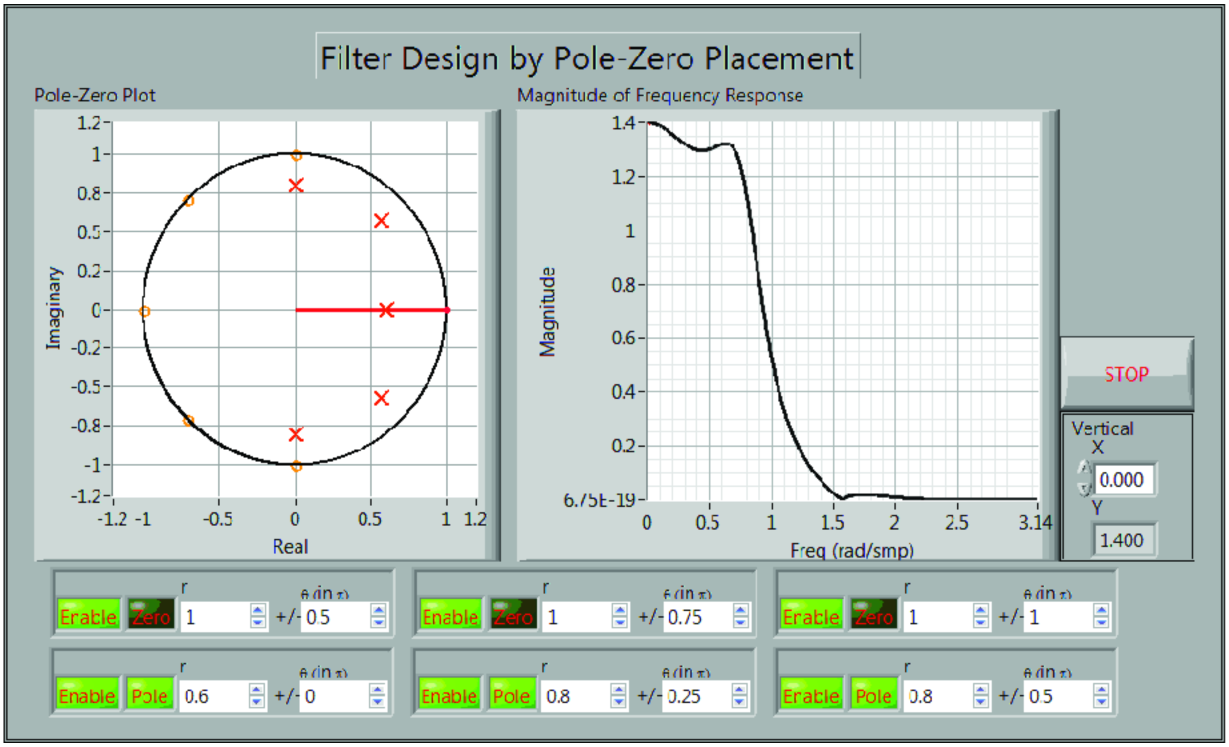
Exercise 8-2 Use LabVIEW Module 8.1 to replicate the result of Section 8-1.2 and produce Fig. 8-3.

Solution:



Exercise 8-3 Use LabVIEW Module 8.1 to replicate Example 8-1 and produce Fig. 8-4.

Solution:



Exercise 8-4 Determine the ARMA difference equation for the notch filter that rejects a 250 Hz sinusoid. The sampling rate is 1000 samples per second. Use $a = 0.99$.

Solution: The discrete-time frequency to be rejected is

$$\Omega_0 = 2\pi \frac{250}{1000} = \frac{\pi}{2}.$$

The notch filter should have zeros at $e^{\pm j\pi/2}$ and poles at $ae^{\pm j\pi/2} = 0.99e^{\pm j\pi/2}$.

The transfer function:

$$\mathbf{H}(\mathbf{z}) = \frac{(\mathbf{z} - e^{j\pi/2})(\mathbf{z} - e^{-j\pi/2})}{(\mathbf{z} - 0.99e^{j\pi/2})(\mathbf{z} - 0.99e^{-j\pi/2})} = \frac{\mathbf{z}^2 + 1}{\mathbf{z}^2 + 0.98} \frac{\mathbf{z}^{-2}}{\mathbf{z}^{-2}} = \frac{1 + \mathbf{z}^{-2}}{1 + 0.98\mathbf{z}^{-2}} = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})}.$$

Cross-multiplying gives

$$\mathbf{Y}(\mathbf{z})(1 + 0.98\mathbf{z}^{-2}) = \mathbf{X}(\mathbf{z})(1 + \mathbf{z}^{-2}).$$

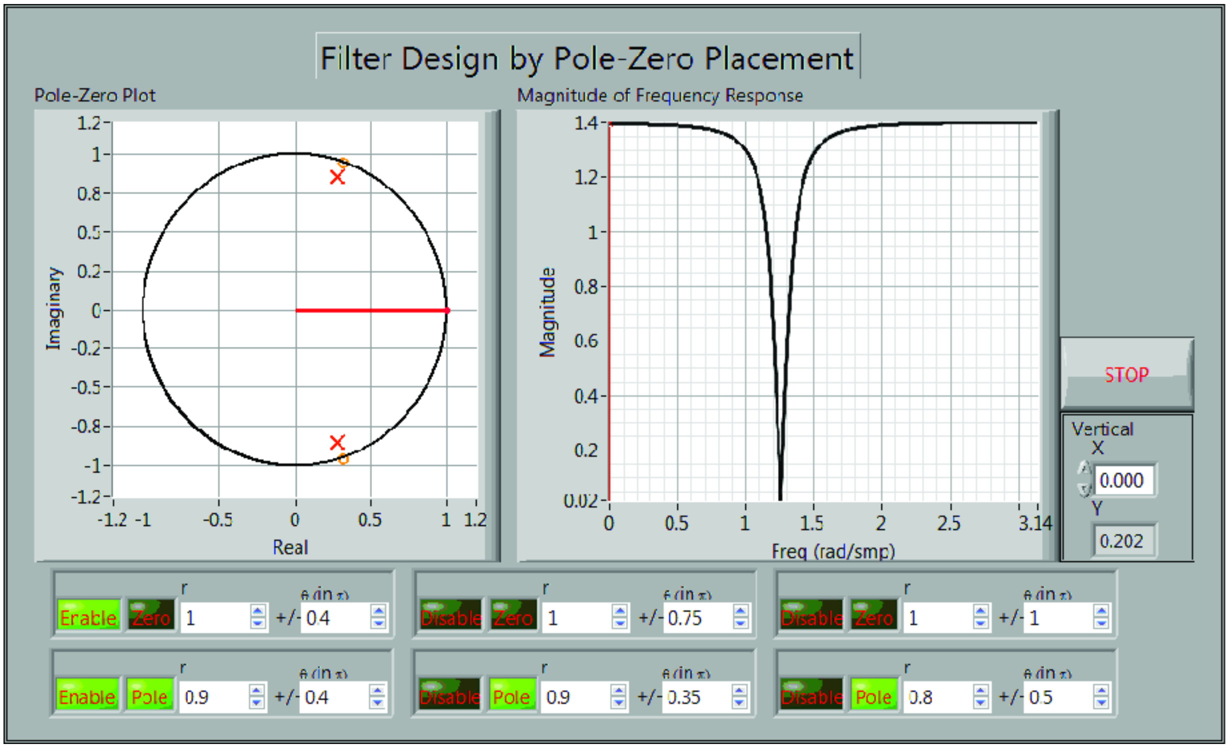
The inverse **z**-transform is

$$\boxed{y[n] + 0.98y[n-2] = x[n] + x[n-2]}.$$

Note that $-2\cos\left(2\pi \frac{250}{1000}\right) = 0$.

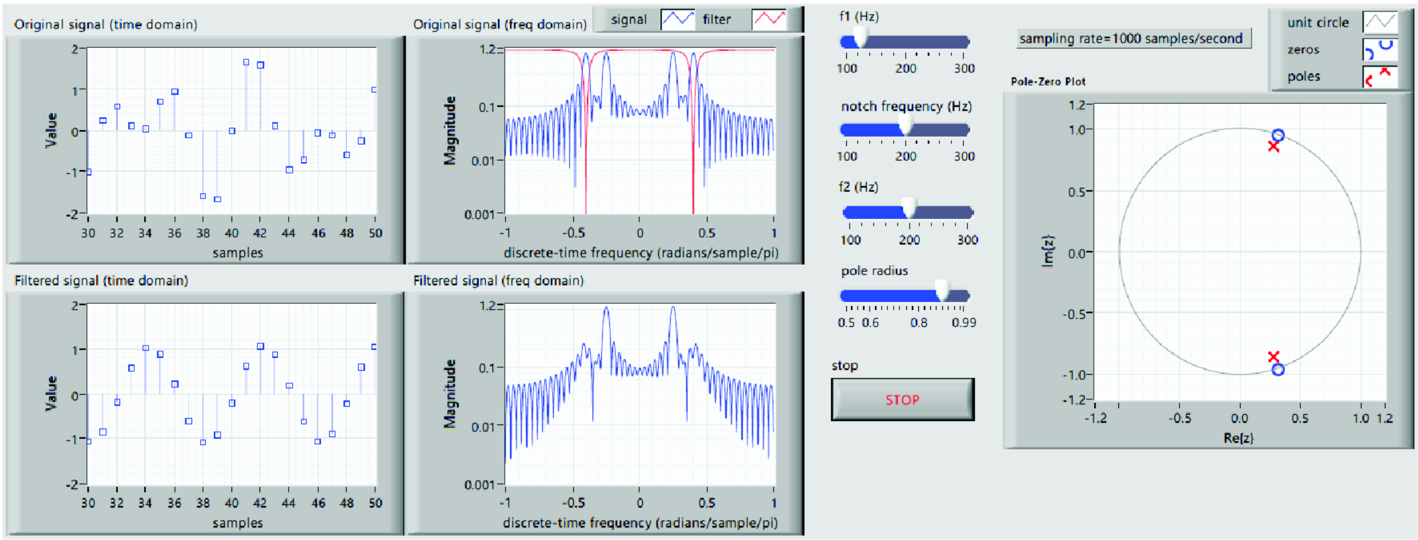
Exercise 8-5 Use LabVIEW Module 8.1 to replicate Example 8-2 and produce the pole-zero and gain plots of Fig. 8-9(a).

Solution:



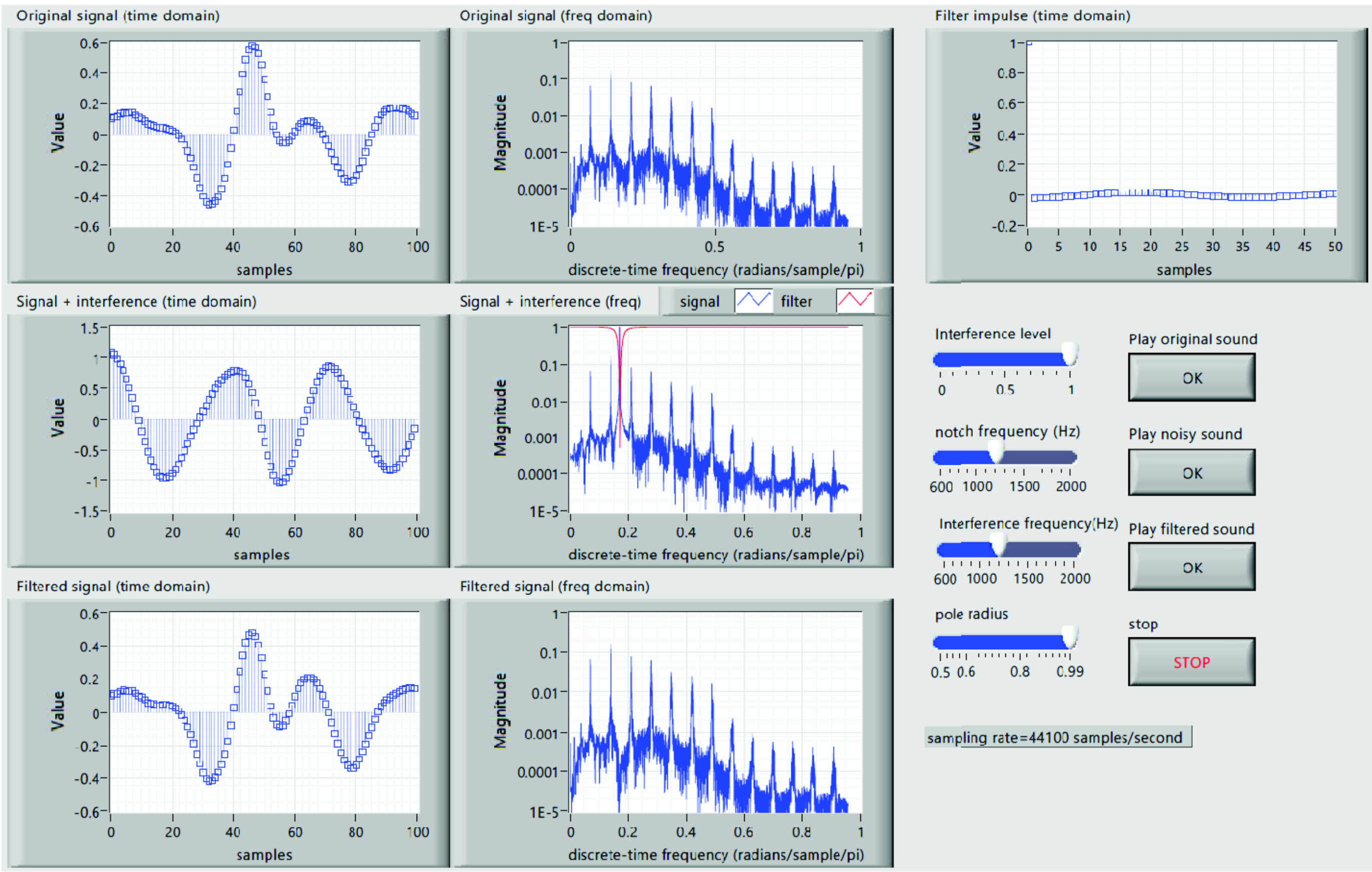
Exercise 8-6 Use LabVIEW Module 8.2 to replicate Example 8-2 and produce the time waveforms of Fig. 8-8.

Solution:



Exercise 8-7 Use LabVIEW Module 8.3 to replicate Example 8-4 and produce the time waveforms of Fig. 8-12 (as stem plots).

Solution:



Exercise 8-8 Determine the ARMA difference equation for a comb filter that rejects periodic interference that has period $= 0.01$ s and is bandlimited to 200 Hz. The sampling rate is 600 samples per second. Use $a = 0.99$.

Solution: Periodic interference with period $= 0.01$ s has a continuous-time Fourier series expansion with harmonics at multiples of 100 Hz. Since the interference is bandlimited to 200 Hz, the interference has harmonics at 100 Hz and 200 Hz. Discrete-time frequencies to be rejected: $2\pi\,\frac{100}{600} = \frac{\pi}{3}$ and $2\pi\,\frac{200}{600} = \frac{2\pi}{3}$. The comb filter should have zeros at $e^{\pm j\pi/3}$ and $e^{\pm j2\pi/3}$, and poles at $ae^{\pm j\pi/3} = 0.99e^{\pm j\pi/3}$ and $ae^{\pm j2\pi/3} = 0.99e^{\pm j2\pi/3}$.

The transfer function is

$$\mathbf{H}(\mathbf{z}) = \frac{(\mathbf{z}-e^{j\pi/3})(\mathbf{z}-e^{-j\pi/3})(\mathbf{z}-e^{j2\pi/3})(\mathbf{z}-e^{-j2\pi/3})}{(\mathbf{z}-0.99e^{j\pi/3})(\mathbf{z}-0.99e^{-j\pi/3})(\mathbf{z}-0.99e^{j2\pi/3})(\mathbf{z}-0.99e^{-j2\pi/3})},$$

which simplifies to

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{z}^4+\mathbf{z}^2+1}{\mathbf{z}^4+0.98\mathbf{z}^2+0.96}\,\frac{\mathbf{z}^{-4}}{\mathbf{z}^{-4}} = \frac{1+\mathbf{z}^{-2}+\mathbf{z}^{-4}}{1+0.98\mathbf{z}^{-2}+0.96\mathbf{z}^{-4}} = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})},$$

since

$$(\mathbf{z}-e^{j\pi/3})(\mathbf{z}-e^{-j\pi/3}) = \mathbf{z}^2 - 2\cos(\pi/3)\mathbf{z}+1 = \mathbf{z}^2-\mathbf{z}+1,$$

and

$$(\mathbf{z}-e^{j2\pi/3})(\mathbf{z}-e^{-j2\pi/3}) = \mathbf{z}^2 - 2\cos(2\pi/3)\mathbf{z}+1 = \mathbf{z}^2+\mathbf{z}+1.$$

Cross-multiplying gives

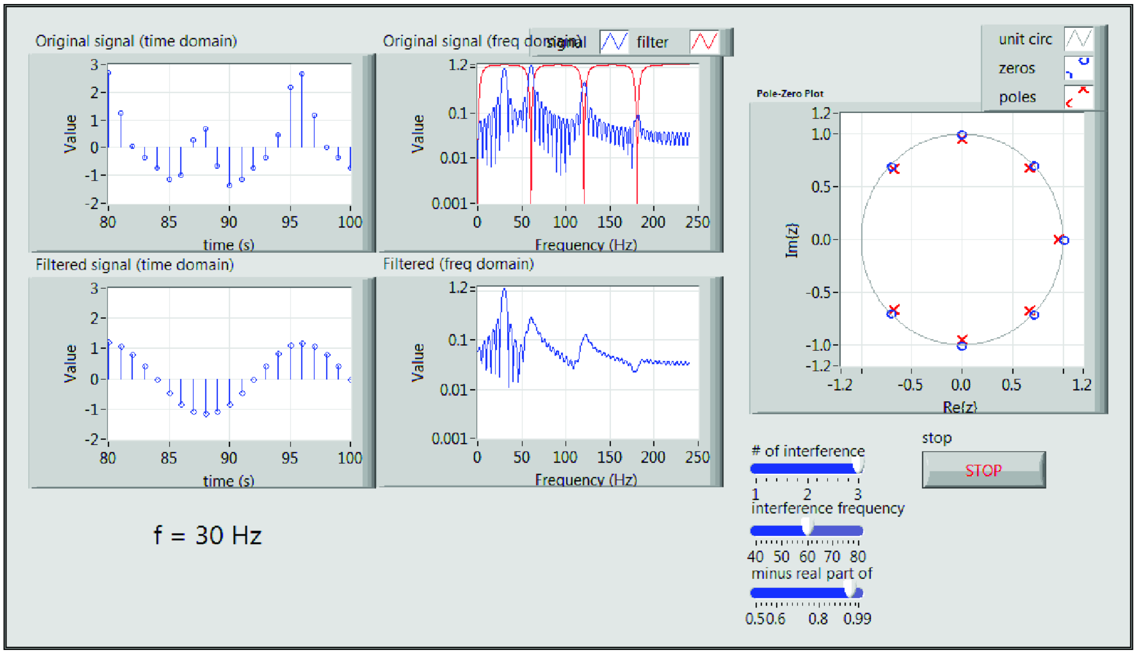
$$\mathbf{Y}(\mathbf{z})\;(1+0.98\mathbf{z}^{-2}+0.96\mathbf{z}^{-4}) = \mathbf{X}(\mathbf{z})\;(1+\mathbf{z}^{-2}+\mathbf{z}^{-4}).$$

The inverse **z**-transform is

$$\boxed{y[n]+0.98y[n-2]+0.96y[n-4] = x[n]+x[n-2]+x[n-4]}.$$

Exercise 8-9 Use LabVIEW Module 8.4 to replicate Example 8-5 and produce the pole-zero and gain plots of Fig. 8-13 and time waveforms of Fig. 8-14.

Solution:



Exercise 8-10 Is deconvolution using real-time signal processing possible for the system: $y[n] = x[n] - 2x[n-1]$?

Solution: Taking the **z**-transform gives

$$\mathbf{Y}(\mathbf{z}) = \mathbf{X}(\mathbf{z}) - 2\mathbf{z}^{-1}\,\mathbf{X}(\mathbf{z}) = (1 - 2\mathbf{z}^{-1})\,\mathbf{X}(\mathbf{z}).$$

The transfer function is

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})} = \frac{1}{1 - 2\mathbf{z}^{-1}}\,\frac{\mathbf{z}}{\mathbf{z}} = \frac{\mathbf{z}}{\mathbf{z} - 2}\ .$$

The system has a zero at 2, and $|2| > 1$, so it is not minimum phase.

The inverse system is not BIBO stable, so real-time deconvolution is $\boxed{\text{not}}$ possible.

Exercise 8-11 A system is given by $y[n] = x[n] - 0.5x[n-1] + 0.4x[n-2]$. What is the difference equation of its inverse system?

Solution: The $\mathbf{z}$-transform is

$$\mathbf{Y}(\mathbf{z}) = \mathbf{X}(\mathbf{z}) - 0.5\mathbf{z}^{-1}\,\mathbf{X}(\mathbf{z}) + 0.4\mathbf{z}^{-2}\,\mathbf{X}(\mathbf{z}) = (1 - 0.5\mathbf{z}^{-1} + 0.4\mathbf{z}^{-2})\,\mathbf{X}(\mathbf{z}).$$

The transfer function is

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})} = \frac{1}{1 - 0.5\mathbf{z}^{-1} + 0.4\mathbf{z}^{-2}}\frac{\mathbf{z}^2}{\mathbf{z}^2} = \frac{\mathbf{z}^2}{\mathbf{z}^2 - 0.5\mathbf{z} + 0.4}\,.$$

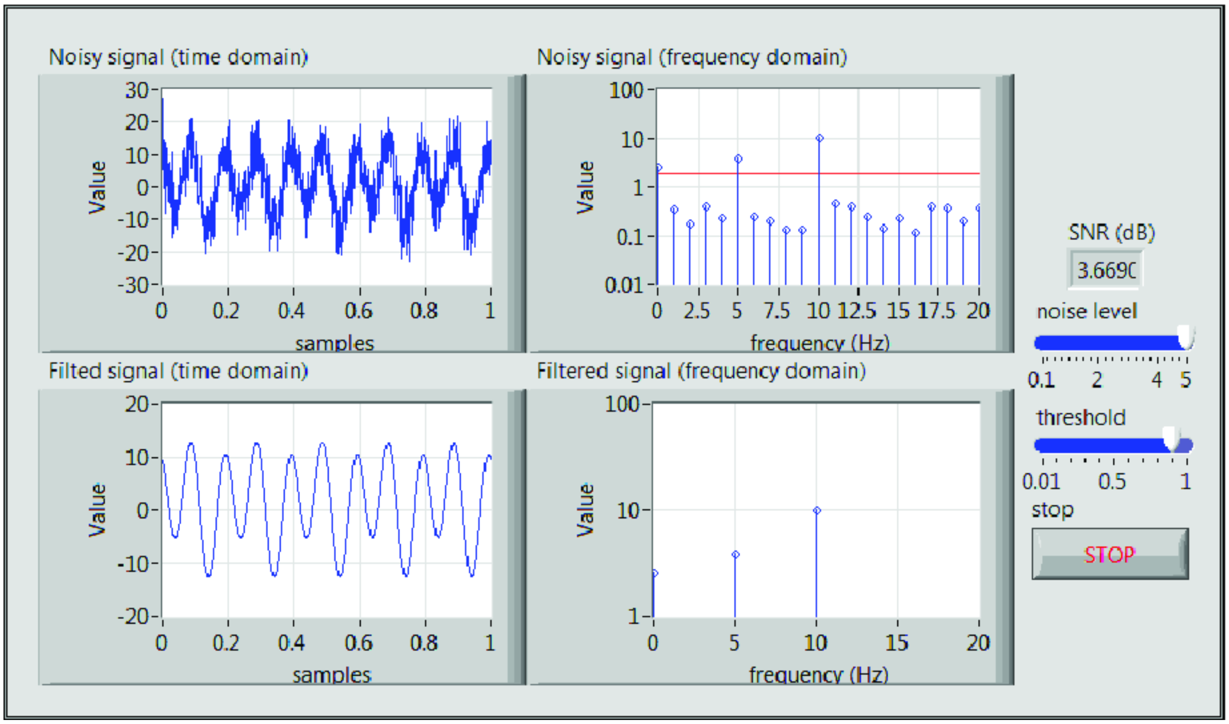
The poles are the roots of denominator $\mathbf{z}^2 - 0.5\mathbf{z} + 0.4 = 0$, which are $\{0.25 \pm j0.581\}$. Since $|0.25 \pm j0.581| = 0.6325 < 1$, the system is minimum phase, and thus it is invertible.

The inverse system is the original system rearranged to

$$\boxed{x[n] = y[n] + 0.5x[n-1] - 0.4x[n-2]}\,.$$

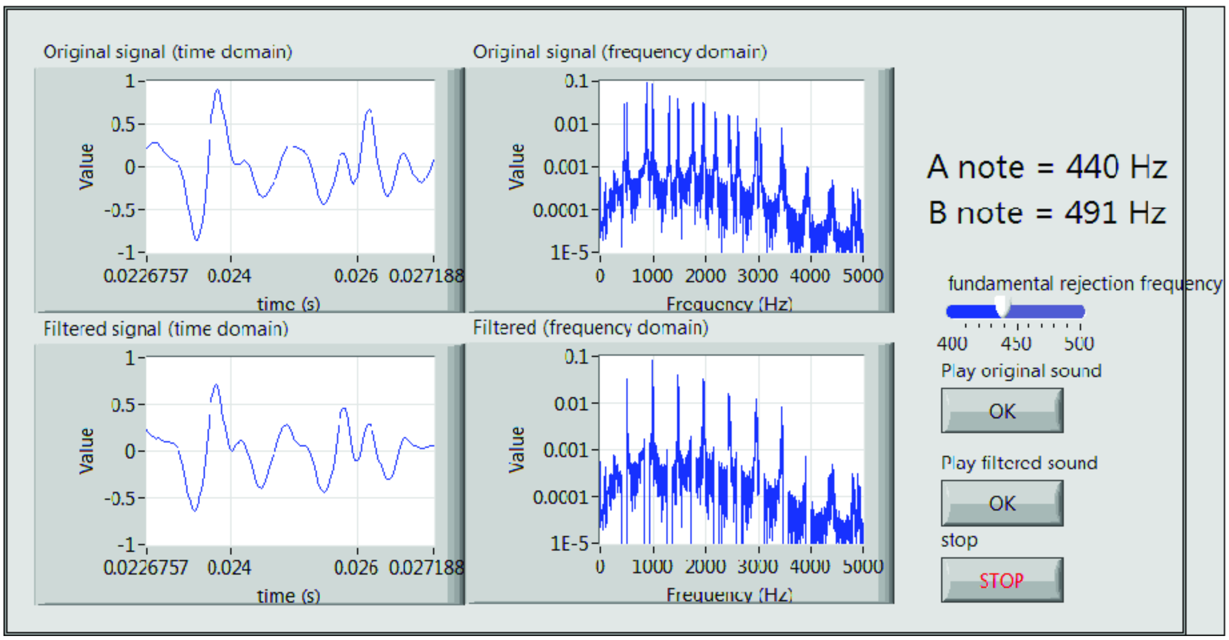
Exercise 8-12 Use LabVIEW Module 8.7 to replicate Example 8-18 and produce the time waveforms and spectra of Fig. 8-18. Note that the dc component is larger.

Solution:



Exercise 8-13 Use LabVIEW Module 8.8 to replicate Example 8-20 and produce the time waveforms and spectra of Fig. 8-20. The time waveforms are different.

Solution:



Exercise 8-14 The spectrum of $\{\cos(0.3\pi n),\ n = 0,\ldots,N-1\}$ is to be computed using the DFT. For what values of N will there be no spectral leakage?

Solution: N must be an integer multiple of the period of $\cos(0.3\pi n)$, which is found from $\frac{2\pi}{0.3\pi} = \frac{20}{3} \Rightarrow$ the period is the numerator 20. $\boxed{N = \text{integer multiple of } 20}$.

Exercise 9-1 Compute the coefficients of a 5-point (a) Bartlett window and (b) Hamming window.

Solution: (a) $\{0, \frac{1}{2}, 1, \frac{1}{2}, 0\}$,
(b) $\{0.08, 0.54, 1.00, 0.54, 0.08\}$.

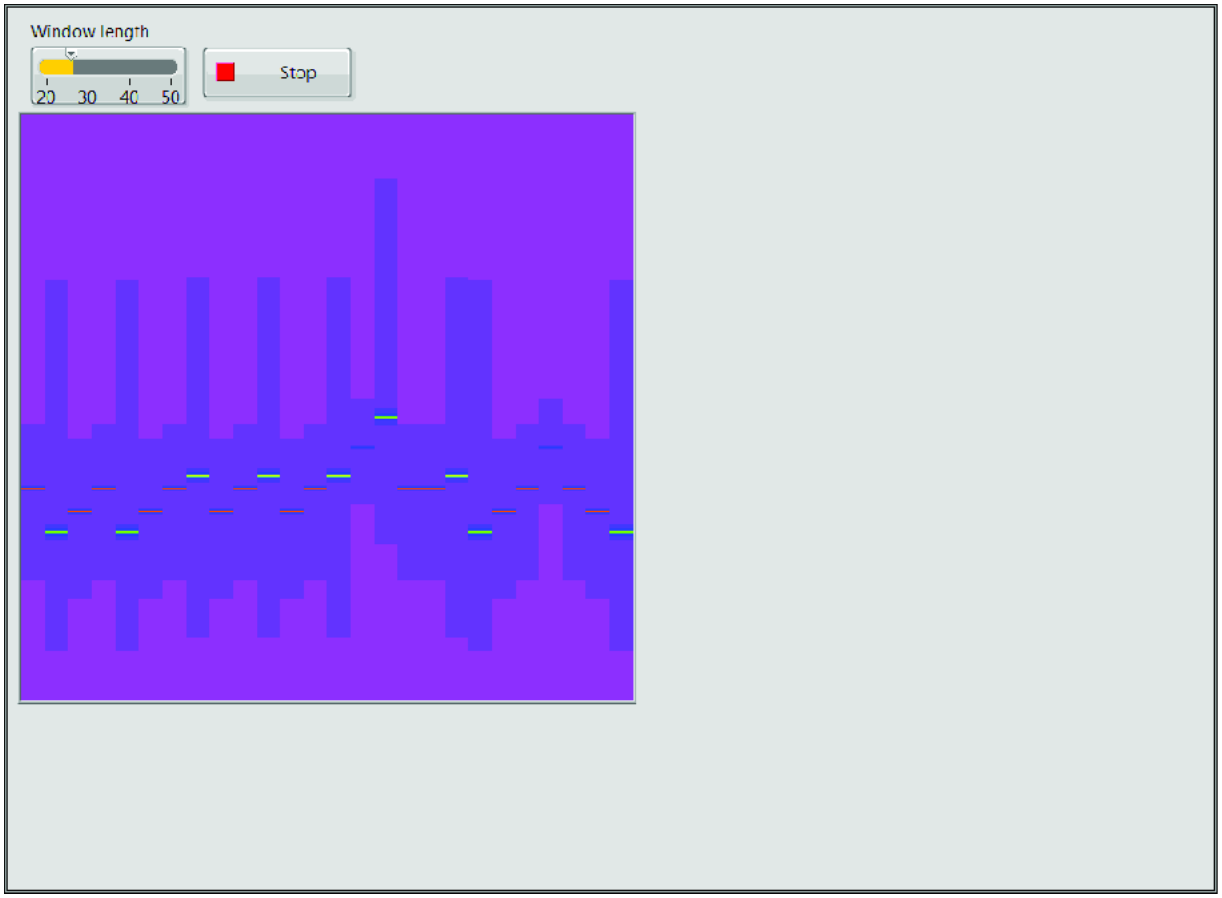
Exercise 9-2 What would the spectrogram of $\cos(t^3)$ look like?

Solution: A parabola, since the instantaneous frequency is

$$f = \frac{1}{2\pi}\frac{dt^3}{dt} = \frac{3}{2\pi}\,t^2.$$

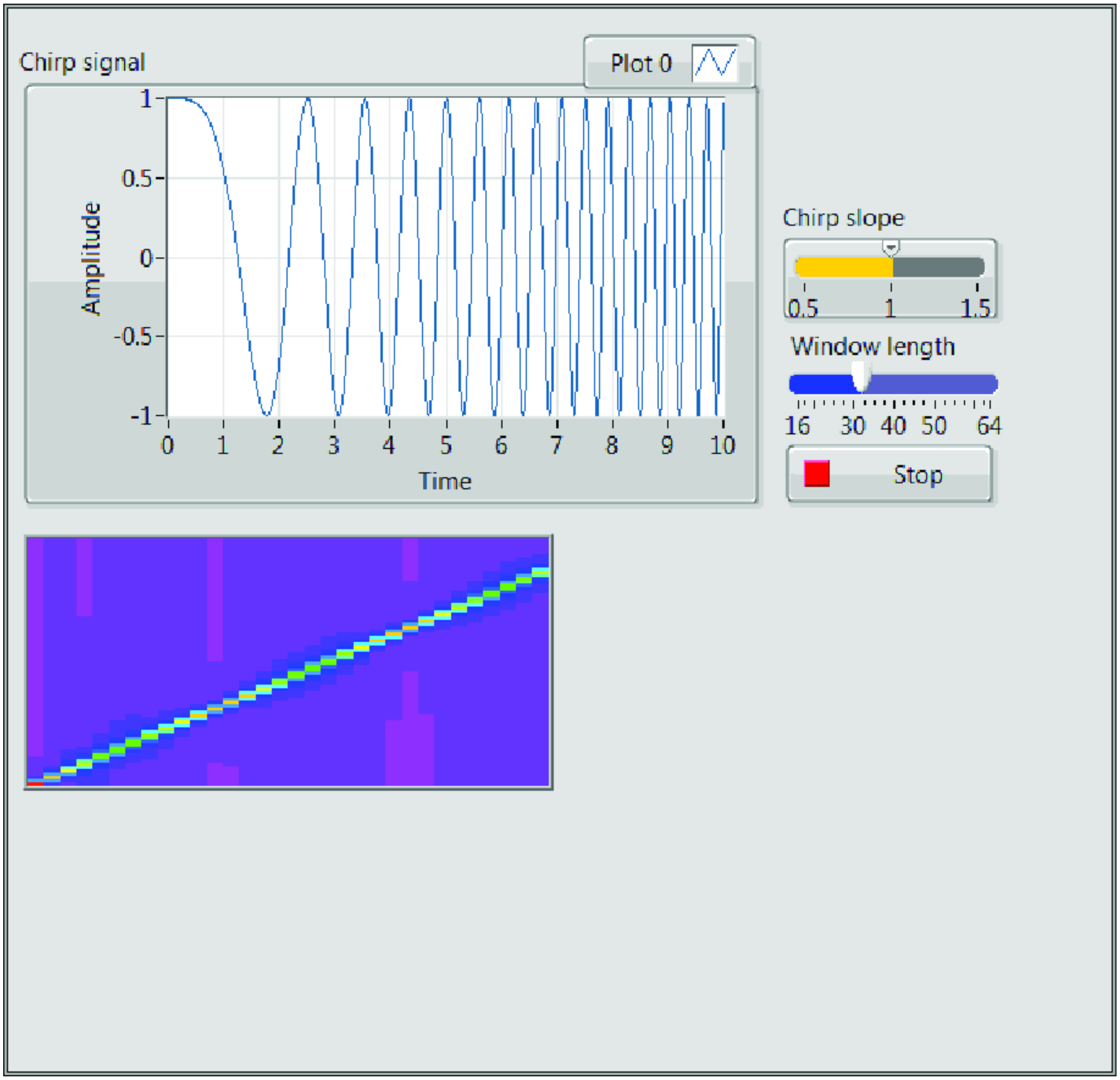
Exercise 9-3 Use LabVIEW Module 9.2 to display the spectrogram of "The Victors." Choose the window length so that the notes do not overlap in time.

Solution:



Exercise 9-4 Use LabVIEW Module 9.3 to display the spectrogram of a chirp with slope 1.0 using window length 32.

Solution:



Exercise 9-5 Design a differentiator of length 3 using a rectangular data window. Interpret your answer.

Solution: $h[n] = \{1, 0, -1\}$ becomes

$$y[n] = x[n+1] - x[n-1],$$

which is a difference operator.

Exercise 9-6 Using the continuous-time filter

$$h_a(t) = \delta(t) - 3e^{-3t}\, u(t)$$

and $T_s = 2$, design a discrete-time filter using impulse invariance.

Solution: The impulse is just feedthrough.

$$h[n] = \delta[n] - T_s\, h_a(nT_s) = \delta[n] - 6e^{-6n}\, u[n].$$

Exercise 9-7 Using the continuous-time filter $\mathbf{H}_{\mathrm{a}}(\mathbf{s}) = \mathbf{s}/(\mathbf{s}+1)$ and $T = 2$, design a discrete-time filter using bilinear transformation.

Solution: Setting $\mathbf{s} = \frac{2}{2}\frac{\mathbf{z}-1}{\mathbf{z}+1}$ in $\mathbf{H}_{\mathrm{a}}(\mathbf{s})$ gives

$$\begin{aligned}\mathbf{H}(\mathbf{z}) &= \frac{(\mathbf{z}-1)/(\mathbf{z}+1)}{1+(\mathbf{z}-1)/(\mathbf{z}+1)}\\ &= \frac{\mathbf{z}-1}{(\mathbf{z}+1)+(\mathbf{z}-1)} = \frac{1}{2}\,(1-\mathbf{z}^{-1}).\end{aligned}$$

So $h[n] = \{\underline{\tfrac{1}{2}}, -\tfrac{1}{2}\}$ is actually FIR here!

Exercise 9-8 We wish to design an IIR discrete-time lowpass filter with cutoff frequency $\Omega_0 = \frac{\pi}{2}$ using bilinear transformation with $T = 0.001$. Determine the continuous-time lowpass filter cutoff frequency ω.

Solution:

$$\omega = \frac{2}{T}\tan\left(\frac{\Omega_0}{2}\right) = \frac{2}{0.001}\tan\left(\frac{\pi/2}{2}\right)$$
$$= 2000\tan\left(\frac{\pi}{4}\right) = 2000.$$

Exercise 9-9 Using bilinear transformation with $T = 0.1$, the continuous-time frequency $\omega = 20$ maps to what discrete-time frequency?

Solution:

$$20 = \omega = \frac{2}{0.1} \tan\left(\frac{\Omega}{2}\right) \quad \Rightarrow \quad 1 = \tan\left(\frac{\Omega}{2}\right)$$

$$\Rightarrow \quad \Omega = \frac{\pi}{2}\,.$$

Exercise 9-10 Use bilinear transformation with $T = 2$ to design an IIR ideal differentiator.

Solution: From Chapter 3, $\mathbf{H}_{\mathrm{a}}(\mathbf{s})=\mathbf{s}$, $\mathbf{s} = \frac{2}{2}\frac{\mathbf{z}-1}{\mathbf{z}+1}$. So

$$\mathbf{H}(\mathbf{z}) = \frac{\mathbf{z}-1}{\mathbf{z}+1} = \frac{\mathbf{Y}(\mathbf{z})}{\mathbf{X}(\mathbf{z})} \quad \Rightarrow$$

$$y[n] + y[n-1] = x[n] - x[n-1].$$

Exercise 9-11 $\cos(0.6\pi n) \rightarrow \boxed{\downarrow 3} \rightarrow ?$

Solution: $\cos(0.6\pi n)$. $\omega = 0.6\pi$ becomes $\omega = 1.8\pi$, which aliases to $\omega = 0.2\pi$, since $1.8\pi \equiv -0.2\pi \equiv 0.2\pi$.

Exercise 9-12 $\cos(0.8\pi n)$ → $\boxed{\downarrow 4}$ → ?

Solution: $\cos(0.8\pi n)$. $\omega = 0.8\pi$ becomes $\omega = 3.2\pi$, which aliases to $\omega = 0.8\pi$, since $3.2\pi \equiv -0.8\pi \equiv 0.8\pi$.

Exercise 9-13 $\cos(0.4\pi n) \rightarrow \boxed{\uparrow 4} \rightarrow ?$

Solution: $\omega = \{0.4\pi, (2-0.4)\pi, (2+0.4)\pi, (4-0.4)\pi\}$ become

$$\omega = \{0.1\pi, 0.4\pi, 0.6\pi, 0.9\pi\}.$$

The input was a single sinusoid, but the output is four sinusoids.

Exercise 9-14 $\cos(0.8\pi n) \rightarrow \boxed{\uparrow 4} \rightarrow$?

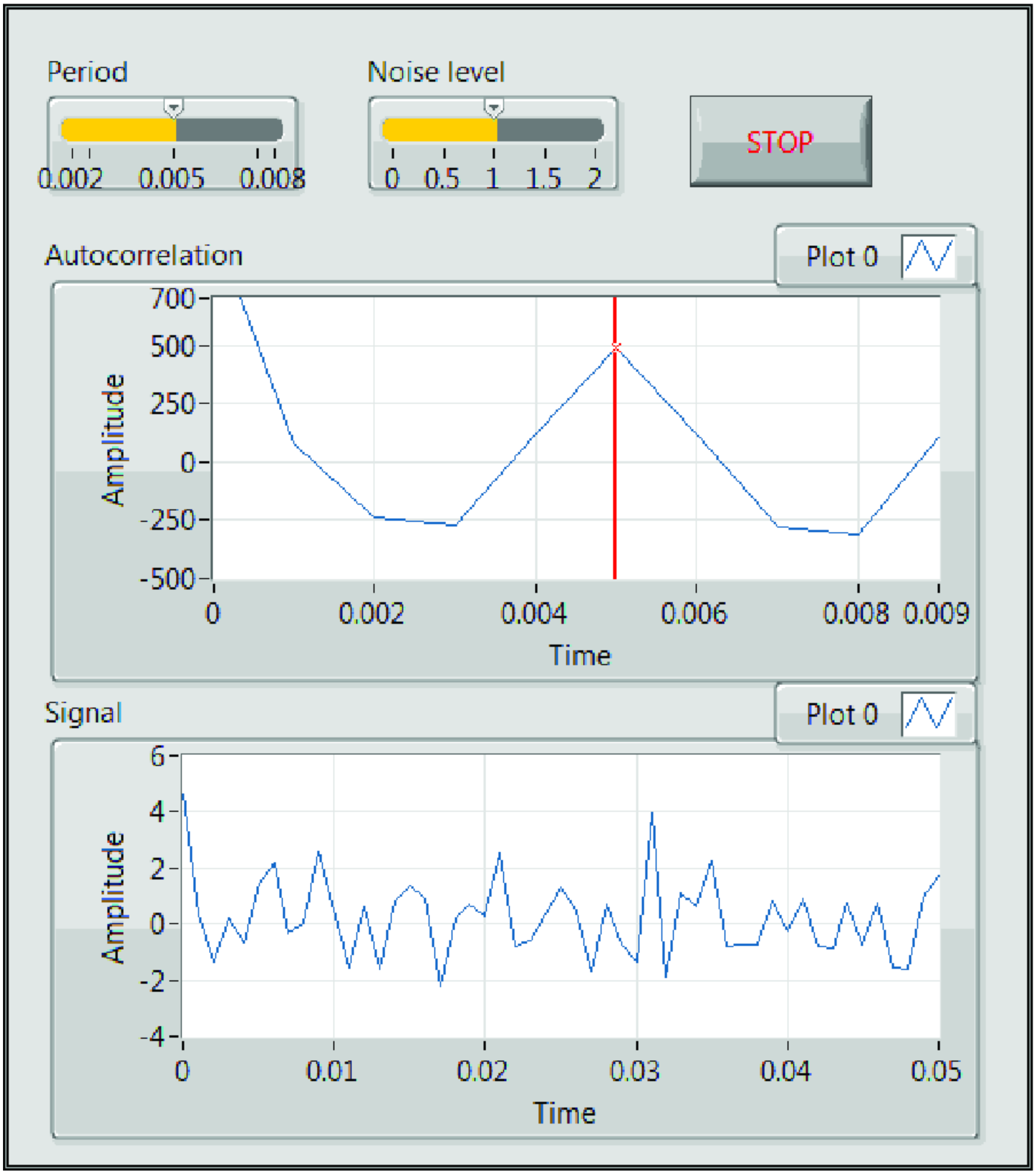
Solution: $\omega = \{0.8\pi, (2-0.8)\pi, (2+0.8)\pi, (4-0.8)\pi\}$ become

$$\omega = \{0.2\pi, 0.3\pi, 0.7\pi, 0.8\pi\}.$$

The input was a single sinusoid, but the output is four sinusoids.

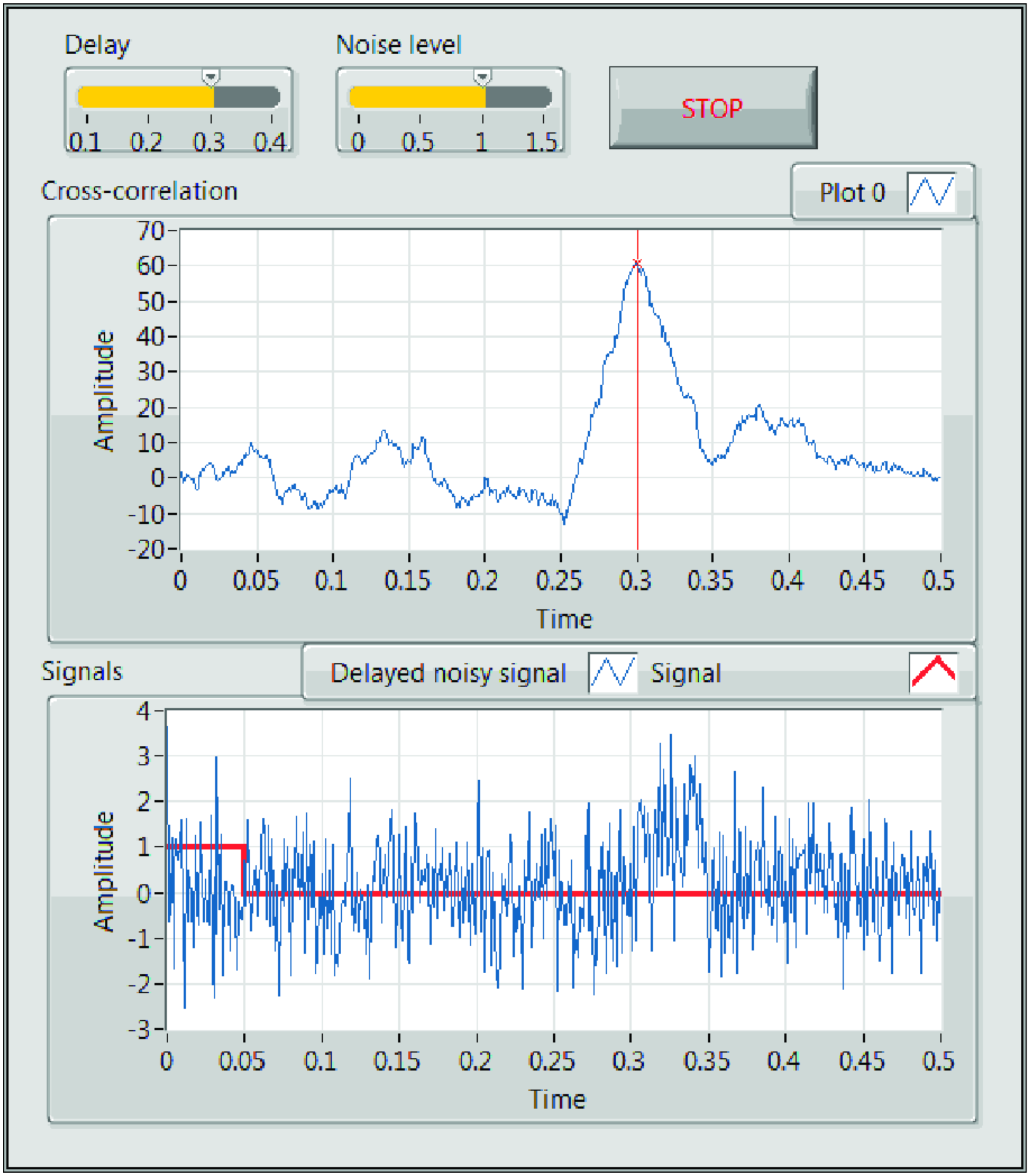
Exercise 9-15 Use LabVIEW Module 9.4 to estimate the period of the waveform with period 0.005 and noise level 1.

Solution:



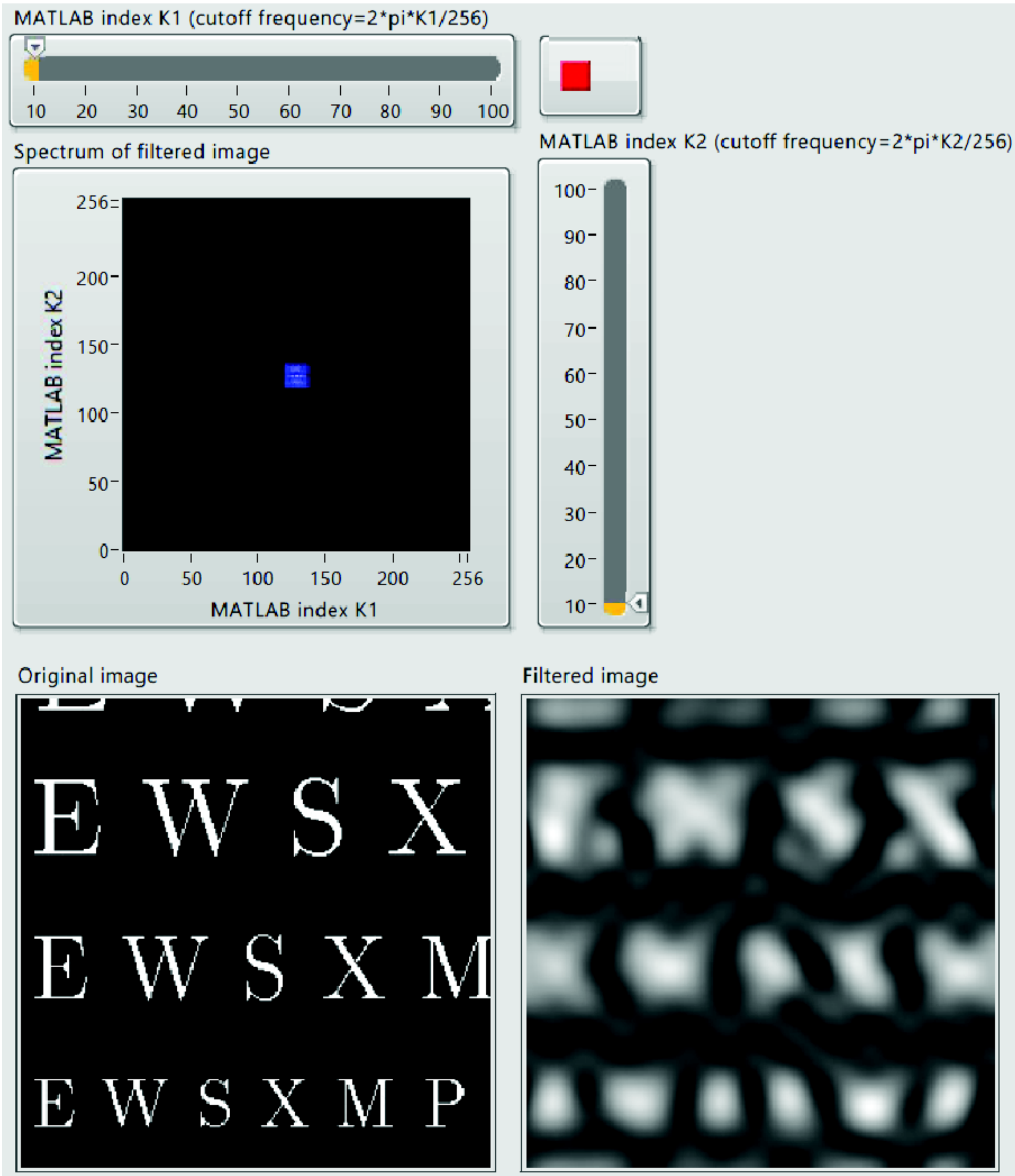
Exercise 9-16 Use LabVIEW Module 9.5 to estimate the time delay of the signal when its actual delay is 0.3 and the noise level is 1.

Solution:



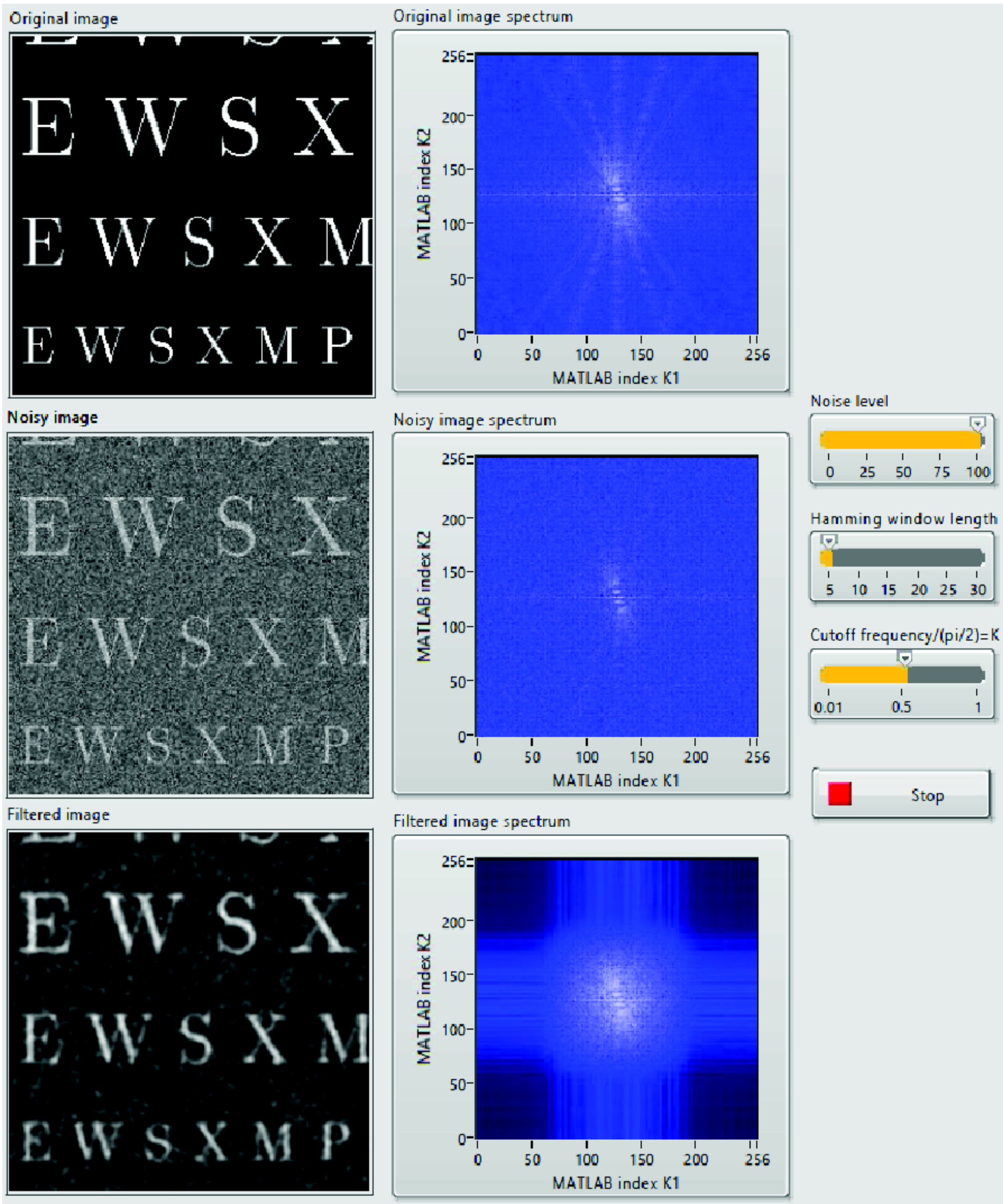
Exercise 10-1 Use LabVIEW Module 10.1 to show the effect of drastic lowpass filtering on the letters image. Set both slides to their minimum values.

Solution:



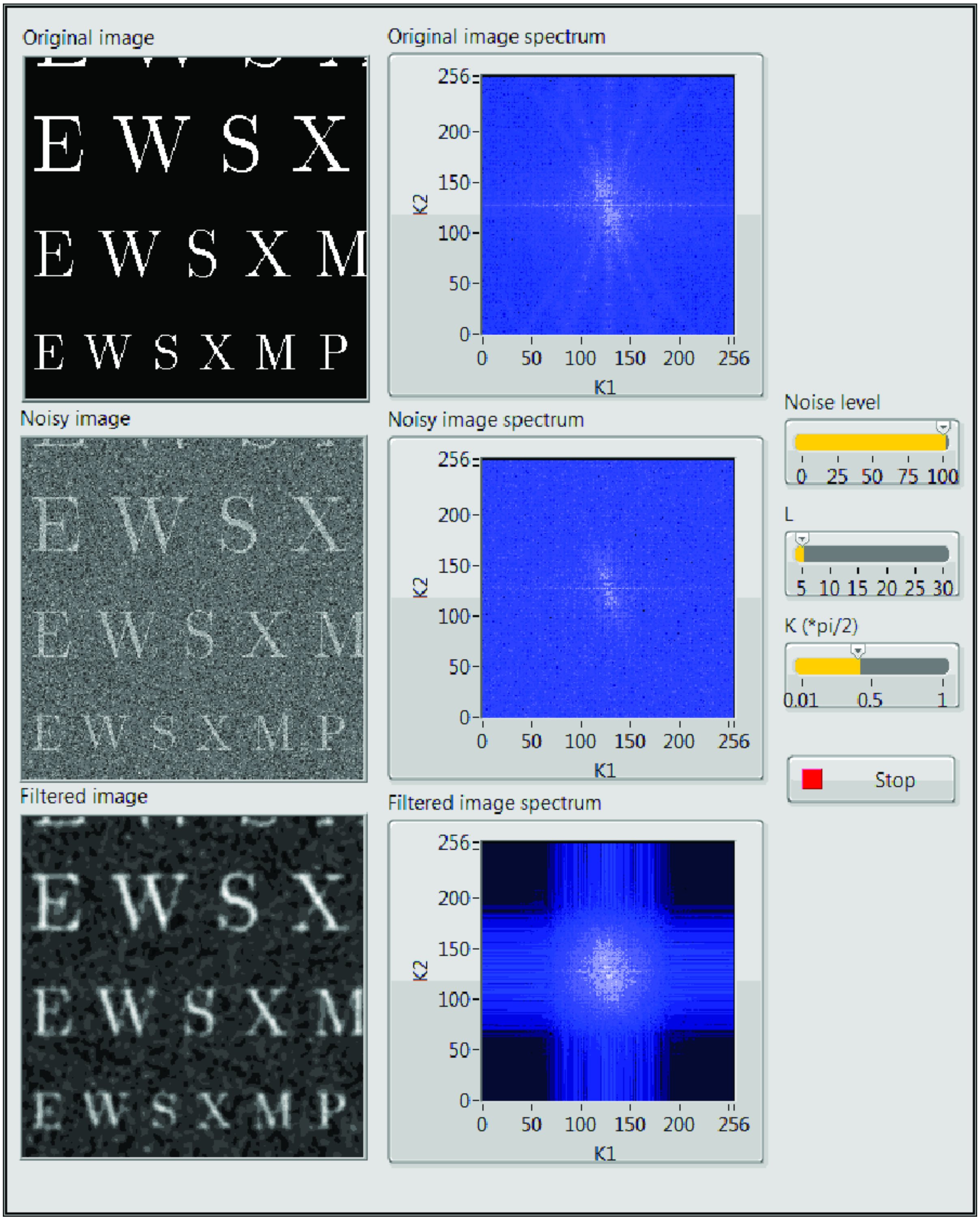
Exercise 10-2 Use LabVIEW Module 10.2 to denoise the letters image using a lowpass filter. Set “K” to 0.5, L to 5, and noise level to 100.

Solution:



Exercise 10-3 Use LabVIEW Module 10.3 to deconvolve the letters image from a noisy blurred version of it. Set the noise level to 1000 and L to 1.

Solution:



Exercise 10-4 Show that $\tilde{g}_{\text{haar}}[n]$ and $\tilde{h}_{\text{haar}}[n]$ are energy-normalized functions.

Solution: From Eq. (10.68) we have

$$(1/\sqrt{2})^2 + (1/\sqrt{2})^2 = (1/\sqrt{2})^2 + (-1/\sqrt{2})^2 = 1.$$

Exercise 10-5 Show that $\mathscr{H}^{-1} = \mathscr{H}^T$ for Eq. (10.89).

Solution: The product $\mathscr{H}\mathscr{H}^T = I$; hence $\mathscr{H}^T = \mathscr{H}^{-1}$.

Exercise 10-6 Show that the normalized Haar scaling function $\tilde{g}_{\text{haar}}[n]$ in Eq. (10.68) satisfies the Smith-Barnwell condition given by Eq. (10.113).

Solution: From Eq. (10.57a) we have

$$\begin{aligned}|\tilde{G}_{haar}(e^{j\omega})| &= |(1+e^{-j\omega})/\sqrt{2}| \\ &= \sqrt{2}|\cos(\omega/2)|\cdot|e^{-j\omega/2}| \\ &= \sqrt{2}|\cos(\omega/2)|\end{aligned}$$

and $|\tilde{G}_{\text{haar}}(e^{j(\omega+\pi)})| = \sqrt{2}|-\sin(\omega/2)|$. Then

$$|G(e^{j\omega})|^2 + |G(e^{j(\omega+\pi)})|^2 = 2\cos^2(\omega/2) + 2\sin^2(\omega/2) = 2.$$

Exercise 10-7 Show that Eq. (10.109) with $L = 1$ holds for the normalized Haar scaling and wavelet basis functions in Eq. (10.68).

Solution: $\tilde{g}_{\text{haar}}[n] = [\underline{1}, 1]/\sqrt{2}$ implies $(-1)^n\, g[1-n] = [\underline{1}, -1]/\sqrt{2} = \tilde{h}_{\text{haar}}[n]$.

Exercise 10-8 Show that if $g[n]$ satisfies the Smith-Barnwell condition and $h[n]$ is determined from $g[n]$ using Eq. (10.109), then $h[n]$ satisfies the Smith-Barnwell condition.

Solution: Scaling function $g[n]$ satisfies the Smith-Barnwell condition Eq. (10.108) if its $\mathbf{z}$-transform $\mathbf{G}(\mathbf{z})$ satisfies $\mathbf{G}(\mathbf{z})\,\mathbf{G}(1/\mathbf{z})+\mathbf{G}(-\mathbf{z})\,\mathbf{G}(-1/\mathbf{z})=2$.

The wavelet function $h[n]$ is determined from the scaling function $g[n]$ using Eq. (10.109), which is $h[n]=(-1)^n\,g[L-n]$. The $\mathbf{z}$-transform of Eq. (10.109) is Eq. (10.102), which is $\mathbf{H}(\mathbf{z})=-\mathbf{G}(-1/\mathbf{z})\,\mathbf{z}^{-L}$. Replacing $\mathbf{z}$ with $1/\mathbf{z}$ gives Eq. (10.103), which is $\mathbf{H}(1/\mathbf{z})=-\mathbf{G}(-\mathbf{z})\,\mathbf{z}^{L}$, and replacing $\mathbf{z}$ with $-\mathbf{z}$ in Eq. (10.102) gives Eq. (10.105), which is $\mathbf{H}(-\mathbf{z})=\mathbf{G}(1/\mathbf{z})\,\mathbf{z}^{-L}$, since L is odd. Replacing $\mathbf{z}$ with $1/\mathbf{z}$ gives (new) $\mathbf{H}(-1/\mathbf{z})=\mathbf{G}(\mathbf{z})\,\mathbf{z}^{L}$, again since L is odd.

Using all of these gives

$$\mathbf{H}(\mathbf{z})\,\mathbf{H}(1/\mathbf{z})=\mathbf{G}(-\mathbf{z}\,)\mathbf{G}(-1/\mathbf{z})$$

and

$$\mathbf{H}(-\mathbf{z})\,\mathbf{H}(-1/\mathbf{z})=\mathbf{G}(1/\mathbf{z})\,\mathbf{G}(\mathbf{z}).$$

Note that $\mathbf{z}^{L}\mathbf{z}^{-L}=1$. So

$$\mathbf{H}(\mathbf{z})\,\mathbf{H}(1/\mathbf{z})+\mathbf{H}(-\mathbf{z})\,\mathbf{H}(-1/\mathbf{z})=\mathbf{G}(-\mathbf{z})\,\mathbf{G}(-1/\mathbf{z})+\mathbf{G}(1/\mathbf{z})\,\mathbf{G}(\mathbf{z})=2,$$

so $\mathbf{H}(\mathbf{z})$, as well as $\mathbf{G}(\mathbf{z})$, satisfies the Smith-Barnwell condition.

Exercise 10-9 Show that D1 Daubechies scaling function $g[n]$ is the normalized Haar scaling function $\tilde{g}_{\text{haar}}[n]$.

Solution: Eq. (10.39) is $G_{\text{haar}}(\mathbf{z}) = (1+\mathbf{z}^{-1})Q$ for constant Q. Inserting into Eq. (10.140) gives $Q = \frac{1}{\sqrt{2}}$.

Exercise 10-10 Show that D1 Daubechies scaling function $g[n]$ is orthogonal to even-valued translations of $g[n]$.

Solution: From Table #1,

$$\begin{aligned}\sum g[n]\, g[n+2] &= g[0]\, g[2] + g[1]\, g[3] \\ &= (.4830)(.2241) + (.8365)(-.1294) = 0.\end{aligned}$$

$\sum g[n]\, g[n+4] = 0$ since $g[n]$ has duration 4. $h[n]$ is also orthogonal to even-valued translations.

Exercise 10-11 Show that a system with two zeros at $\mathbf{z} = 1$ compresses signals linear in time n to zero.

Solution: If $H(z)$ has two zeros at $\mathbf{z} = 1$, it must have the form

$$H(\mathbf{z}) = (\mathbf{z}-1)^2 P(\mathbf{z}) = (\mathbf{z}^2 - 2\mathbf{z} + 1)\ P(\mathbf{z}).$$

Let $x[n] = an + b$ for constants a and b.

$$\begin{aligned} x[n] * h[n] &= x[n] * \{1, -2, \underline{1}\} * p[n] \\ &= (x[n+2] - 2x[n+1] + x[n]) * p[n] \\ &= 0 * p[n] = 0 \end{aligned}$$

since we have

$$x[n+2] - 2x[n+1] + x[n] = (a(n+2)+b) - 2(a(n+1)+b) + (an+b) = 0.$$

Exercise 10-12 Show that for separable 2-D scaling and wavelet functions, the 2-D Smith-Barnwell condition Eq. (10.159) is satisfied if the 1-D Smith-Barnwell condition given by Eq. (10.113) is satisfied.

Solution: Inserting Eq. (10.160) into Eq. (10.159) gives

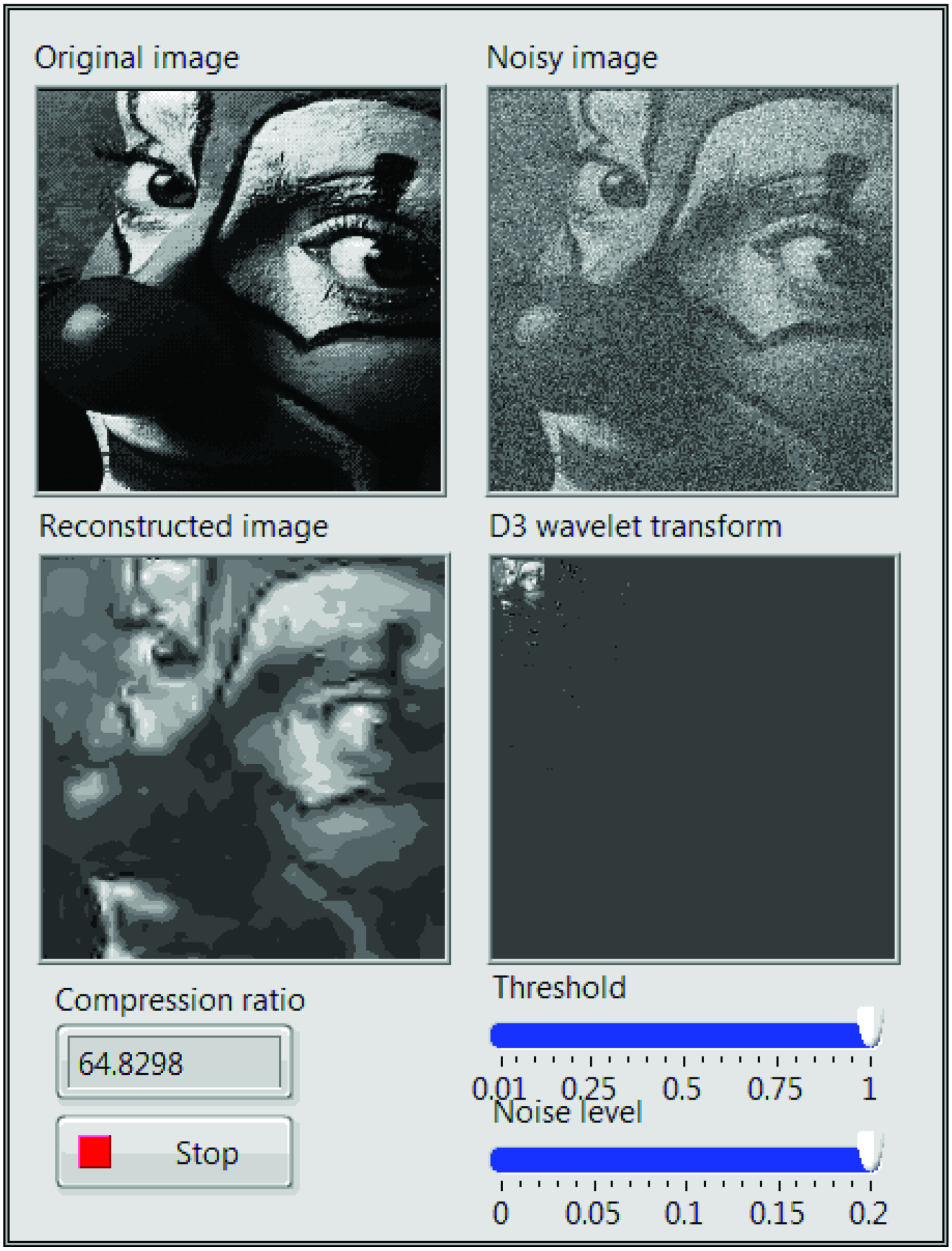
$$\begin{aligned}
&|G(e^{j\omega_1})\, G(e^{j\omega_2})|^2 + |G(e^{j(\omega_1+\pi)})\, G(e^{j(\omega_2+\pi)})|^2 \\
&\qquad + |G(e^{j(\omega_1+\pi)})\, G(e^{j\omega_2})|^2 + |G(e^{j\omega_1})\, G(e^{j(\omega_2+\pi)})|^2 \\
&= \left(|G(e^{j\omega_1})|^2 + |G(e^{j(\omega_1+\pi)})|^2\right) |G(e^{j\omega_2})|^2 \\
&\qquad + \left(|G(e^{j\omega_1})|^2 + |G(e^{j(\omega_1+\pi)})|^2\right) |G(e^{j(\omega a_2+\pi)})|^2 \\
&= 2|G(e^{j\omega_2})|^2 + 2|G(e^{j(\omega_2+\pi)})|^2 = 4.
\end{aligned}$$

Exercise 10-13 Use LabVIEW Module 10.5 to compress and then decompress the clown image. Use a threshold of 0.5. What compression ratio does this produce?

Solution: 52.1512.

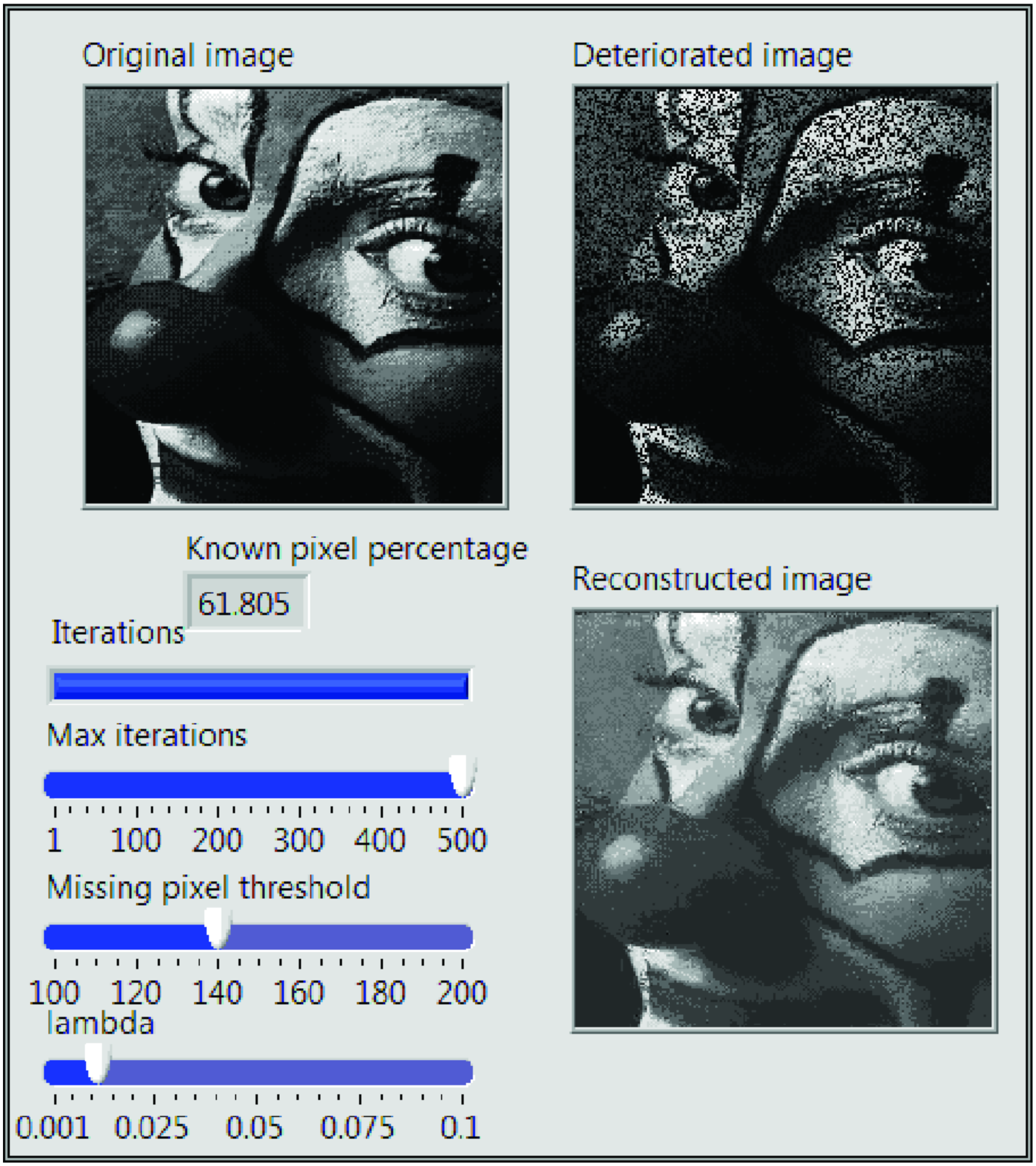
Exercise 10-14 Use LabVIEW Module 10.6 to denoise the clown image. Use a noise level of 0.2 and threshold of 1. Discuss the result.

Solution:



Exercise 10-15 Use LabVIEW Module 10.7 to inpaint the clown image. Use lambda = 0.01, missing pixel threshold = 140, and max iterations = 500.

Solution:



Exercise B-1 Express the following complex functions in polar form:

$$\mathbf{z}_1 = (4 - j3)^2,$$
$$\mathbf{z}_2 = (4 - j3)^{1/2}.$$

Solution:

$$\begin{aligned}
\mathbf{z}_1 &= (4 - j3)^2 \\
&= [\sqrt[+]{4^2 + 3^2}\, e^{-j\tan^{-1} 3/4}]^2 = (5e^{-j36.87^\circ})^2 = 25e^{-j73.74^\circ} \\
\mathbf{z}_2 &= (4 - j3)^{1/2} \\
&= \left[\sqrt[+]{4^2 + 3^2}\, e^{-j\tan^{-1} 3/4}\right]^{1/2} \\
&= \pm\sqrt{5}\, e^{-j18.43^\circ}.
\end{aligned}$$

Exercise B-2 Show that $\sqrt{2j} = \pm(1+j)$.

Solution:

$$\begin{aligned}
\sqrt{2j} &= \sqrt{2e^{j90^\circ}} \\
&= \pm\sqrt{2}\, e^{j45^\circ} \\
&= \pm\sqrt{2}\left(\frac{\cos 45^\circ + j\sin 45^\circ}{2}\right) \\
&= \pm\sqrt{2}\left(\frac{\sqrt{2}+j\sqrt{2}}{2}\right) = \pm(1+j).
\end{aligned}$$